Solution (#324) We shall prove the following more general result first: if $p_1, \ldots, p_n$ and $x_1, \ldots, x_n$ are reals such that

$$p_1 + p_2 + \cdots + p_n = 1, \quad \text{and} \quad p_i \geqslant 0 \text{ for each } i,$$

then

$$(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)^2 \leqslant p_1 (x_1)^2 + p_2 (x_2)^2 + \cdots + p_n (x_n)^2.$$

Note that

$$\begin{aligned}
\left(\sum_{i=1}^{n} p_i (x_i)^2\right) - \left(\sum_{i=1}^{n} p_i x_i\right)^2 &= \left(\sum_{i=1}^{n} p_i (1-p_i)(x_i)^2\right) - 2\sum_{i<j} p_i p_j x_i x_j \\
&= \left(\sum_{i=1}^{n} p_i \left(\sum_{j \neq i} p_j\right)(x_i)^2\right) - 2\sum_{i<j} p_i p_j x_i x_j \\
&= \left(\sum_{i \neq j} p_i p_j (x_i)^2\right) - 2\sum_{i<j} p_i p_j x_i x_j \\
&= \sum_{i<j} p_i p_j \left((x_i)^2 - 2x_i x_j + (x_j)^2\right) \\
&= \sum_{i<j} p_i p_j \left(x_i - x_j\right)^2 \geqslant 0.
\end{aligned}$$

Suppose now that a random variable X takes values $x_1, \ldots, x_n$ with probabilities $p_1, \cdots, p_n$. We have just shown that

$$E(X)^2 = (p_1 x_1 + \cdots + p_n x_n)^2 \leqslant p_1 (x_1)^2 + \cdots + p_n (x_n)^2 = E(X^2).$$

Our desired result follows by taking $X = |N - \mu|$. Further we see from the above proof that we get a strict inequality unless X is constant. For $X = |N - \mu|$, where N is binomially distributed, this means that $p = 0$ or $p = 1$.