Solution (#340) This version of Pascal's triangle is formed by each number in a row being the *difference* of the numbers above it.

$$\begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&0&&1&&& \\ &&1&&1&&1&&1&& \\ &1&&0&&0&&0&&1& \\ 1&&1&&0&&0&&1&&1 \end{array} \qquad \begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&2&&1&&& \\ &&1&&3&&3&&1&& \\ &1&&4&&6&&4&&1& \\ 1&&5&&10&&10&&5&&1 \end{array}$$

Comparing it with the original triangle of Pascal, it is not hard to see – at least up to the $\binom{5}{k}$ row – that the new version contains a 0 where the original contains an even number and a 1 where the original contains an odd number. We can show by induction that this remains the case throughout the triangle: if this were the case in the kth row then the cases that would arise are:

New		Difference	Original		Sum
0	0	0	even	even	even
0	1	1	even	odd	odd
1	0	1	odd	even	odd
1	1	0	odd	odd	even

so that we see the same pattern continues into the $(k+1)$th row.

Note that the rows which are all 1s immediately precede the rows that are all 0s except for the end 1s, and vice versa. We will show that the latter occurs in precisely the nth rows where n is a power of 2 – and so the rows that are all 1s appear in precisely rows (2^r-1)th rows.

As a starting inductive hypothesis we will suppose that

$$(1+x)^{2^r} = 1 + xE_r(x) + x^{2^r}$$

where $E_r(x)$ is a polynomial of degree at most 2^r-2 with even integer coefficients. This is true when $r=1$ with $E(x)=2$, If true in the rth case then

$$\begin{aligned} (1+x)^{2^{r+1}} &= \left((1+x)^{2^r}\right)^2 \\ &= \left(1+xE_r(x)+x^{2^r}\right)^2 \\ &= 1 + x^2E_r(x)^2 + x^{2^{r+1}} + 2xE_r(x) + 2x^{2^r} + 2x^{2^r+1}E(x) \\ &= 1 + x\left(xE_r(x)^2 + 2E_r(x) + 2x^{2^r-1} + 2x^{2^r}E(x)\right) + x^{2^{r+1}} \end{aligned}$$

where

$$E_{r+1}(x) = xE_r(x)^2 + 2E_r(x) + 2x^{2^r-1} + 2x^{2^r}E(x),$$

is a polynomial with even coefficients of degree at most

$$\max\{1+2(2^r-2), 2(2^r-2), 2^r-1, 2^r+2^r-2\} = 2^{r+1}-2.$$

The result follows by induction.

Now say that n is not a power of 2. We may write

$$n = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_k}$$

where $r_1 < r_2 < \cdots < r_k$ by expressing n in binary. So

$$\begin{aligned} (1+x)^n &= (1+x)^{2^{r_1}}(1+x)^{2^{r_2}} \times \cdots \times (1+x)^{2^{r_k}} \\ &= (1+xE_{r_1}(x)+x^{2^{r_1}})(1+xE_{r_2}(x)+x^{2^{r_2}}) \times \cdots \times (1+xE_{r_k}(x)+x^{2^{r_k}}). \end{aligned}$$

When expanding this we can see that any term involving an $E_{r_i}(x)$ will not affect the parity (evenness/oddness) of any coefficients. So we can note that the coefficient of $x^{2^{r_1}}$ is odd as contributions to this coefficient either involve an $E_{r_i}(x)$ term or the $x^{r_1} \times 1 \times 1 \times \cdots \times 1$ term which contributes to the coefficient being odd.

See also the solution to #342 for a more general alternative approach to this exercise.