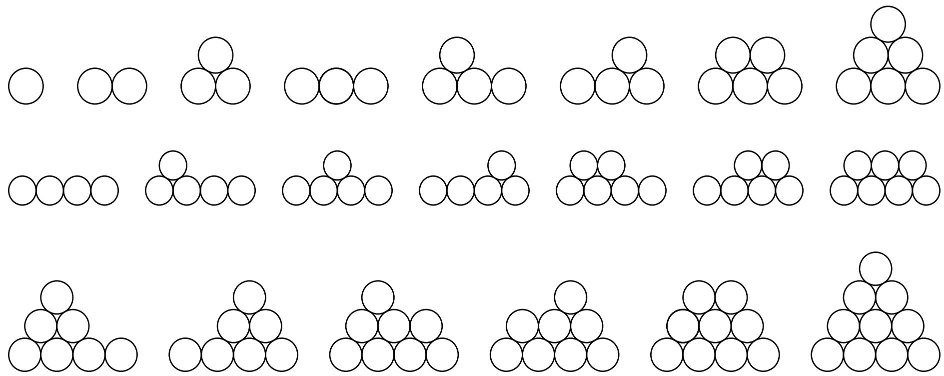
Solution (#358) The diagrams below show the possible block fountains with 4 or fewer disks in the first row.



So

$$b_1 = 1, \qquad b_2 = 2, \qquad b_3 = 5, \qquad b_4 = 13.$$

Consider now the general situation of making a block with k disks in the first row. For each such block fountain the second row and above themselves comprise a block fountain with $k-1$ disk in its 'first' row – the one exception to this is the single row block fountain of k disks.

Conversely, given any block fountain with $i < k$ disks in its first row, we can place that block fountain on top of our first row of k disks in $k-i$ places. Each such positioning leads to a distinct block fountain with k disks in its first row.

Thus we have

$$b_k = \sum_{i=1}^{k-1} (k-i) b_i + 1.$$

Note that $b_1 = F_0 = 1$. Suppose now that $b_i = F_{2i-1}$ for $1 \leqslant i < k$. By the above recurrence we have

$$b_k = \sum_{i=1}^{k-1} (k-i) F_{2i-1} + 1 = k \sum_{i=1}^{k-1} F_{2i-1} - \sum_{i=1}^{k-1} i F_{2i-1} + 1.$$

From #348 we have

$$\sum_{i=1}^{k-1} F_{2i-1} = F_{2k-2},$$

and from #353 we have that

$$\sum_{i=1}^{k-1} i F_{2i-1} = (k-1) F_{2k-2} - F_{2k-3} + 1.$$

Thus

$$\begin{aligned} b_k &= kF_{2k-2} - ((k-1)F_{2k-2} - F_{2k-3} + 1) + 1 \\ &= F_{2k-2} + F_{2k-3} \\ &= F_{2k-1}, \end{aligned}$$

as required and the result follows by induction.