Solution (#359) We define the sequence c_k by

$$c_{2k-1} = b_k, \qquad c_{2k} = b_1 + b_2 + \cdots + b_k,$$

where b_k is as in #358. So we immediately then have

$$c_{2k+2} = (b_1 + b_2 + \cdots + b_k) + b_{k+1} = c_{2k} + c_{2k-1}$$

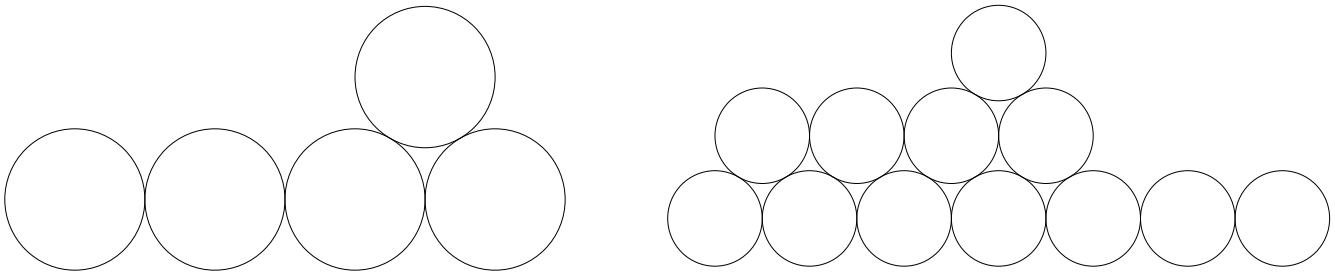
as required.

The term c_{2k+1} denotes the number of block fountains with $k+1$ circles in the bottom row. For each such block fountain there is either a circle in the leftmost position of the second row or not.

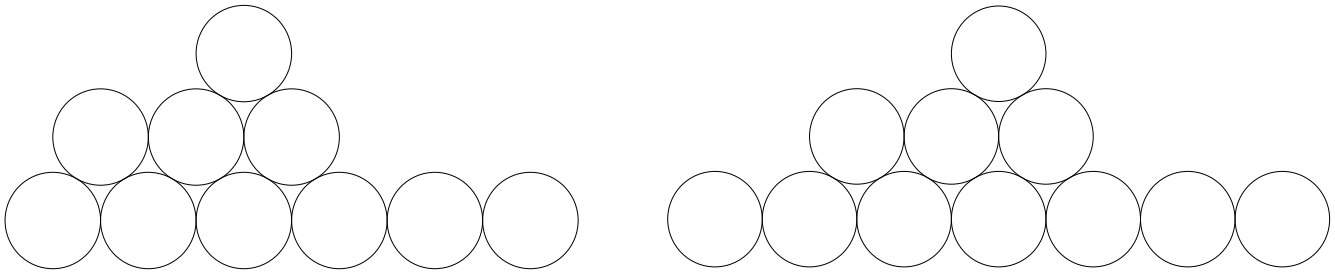
If there is a circle in that leftmost position, then the second row and above can be considered as a block fountain with k or fewer circles in its bottom row, and there are c_{2k} such block fountains.

On the other hand if there is no such circle in the leftmost position of the second row, then the given block fountain can be thought of as a block fountain with k circles in the bottom row with a further circle added on the left of its bottom row. There are c_{2k-1} such block fountains below.

The diagrams below illustrate these ideas:



c_{12} fountains considered within c_{13} fountains:
put your fountain of 6 or fewer base circles and place immediately as the second row of a fountain with 7 base circles.



c_{11} fountains considered within c_{13} fountains:
take your fountain with 6 base circles and add a circle to the left of the base row.

Hence we also have $c_{2k-1} + c_{2k} = c_{2k+1}$.

We then have $c_r + c_{r+1} = c_{r+2}$ for all $r \geqslant 1$ and further have $c_1 = b_1 = 1$ and $c_2 = b_2 = 1$ and hence $c_r = F_r$ for all r.