Solution (#381) Let m and n be positive integers. Say that $m = \sum_i F_{c_i}$ and $n = \sum_j F_{d_j}$ so that

$$m \circ n = \sum_{i,j} F_{c_i + d_j}.$$

The RHS here though is unlikely to be the Zeckendorf representation of $m \circ n$ as it may include consecutive Fibonacci numbers. Say that

$$m \circ n = \sum_{i,j} F_{c_i + d_j} = \sum_k F_{e_k}$$

is its Zeckendorf representation. To get from one sum of Fibonacci numbers to the other involves repeated uses of the recurrences

$$F_k = F_{k-1} + F_{k-2}.$$

However as $\beta^k = \beta^{k-1} + \beta^{k-2}$ also holds for each k, it follows that

$$(m \circ n)_\beta = \sum_k \beta^{e_k} = \sum_{i,j} \beta^{c_i + d_j} = \left(\sum_i \beta^{c_i} \right) \left(\sum_j \beta^{d_j} \right) = m_\beta n_\beta.$$

So given three integers $l, m, n \geqslant 1$ we have

$$((l \circ m) \circ n)_\beta = (l \circ m)_\beta n_\beta = l_\beta m_\beta n_\beta = l_\beta (m \circ n)_\beta = (l \circ (m \circ n))_\beta.$$

By the uniqueness of these expressions (#380(iii)) we conclude that

$$(l \circ m) \circ n = l \circ (m \circ n)$$

and that $\circ$ is associative.