Solution (#385) Firstly note that the given answer can be rewritten as

$$\frac{7-\sqrt{5}}{2} = 3 - \frac{1}{\alpha},$$

where $\alpha = (1+\sqrt{5})/2$ is as defined in Proposition 86. Note that

$$\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_4} = \frac{7}{3} = 3 - \frac{2}{3}; \qquad \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_4} + \frac{1}{F_8} = \frac{50}{21} = 3 - \frac{13}{21},$$

and so it seems reasonable to conjecture that

$$\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_4} + \cdots + \frac{1}{F_{2^n}} = 3 - \frac{F_{2^n-1}}{F_{2^n}}.$$

If this is true then the exercise is completed by letting $n \to \infty$.

The above is true for $n = 1$ as $1/F_1 + 1/F_2 = 2 = 3 - 1/1 = 3 - F_1/F_2$. If our conjecture holds in the nth case then

$$\begin{aligned} & \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_4} + \cdots + \frac{1}{F_{2^n}} + \frac{1}{F_{2^{n+1}}} \\ = & \left(3 - \frac{F_{2^n-1}}{F_{2^n}}\right) + \frac{1}{F_{2^{n+1}}} \\ = & 3 + \left(\frac{1}{F_{2^{n+1}}} - \frac{F_{2^n-1}}{F_{2^n}}\right) \end{aligned}$$

To conclude the proof we desire result

$$\frac{1}{F_{2^{n+1}}} - \frac{F_{2^n-1}}{F_{2^n}} = -\frac{F_{2^{n+1}-1}}{F_{2^{n+1}}},$$

which is equivalent to

$$F_{2^n} = F_{2^{n+1}} F_{2^n-1} - F_{2^{n+1}-1} F_{2^n}$$

which is a special case of d'Ocagne's Identity (#351).