Solution (#419) Suppose that HTH comes first with the last H on the nth toss. This means that neither the sequence HTH, nor the sequence HTT has already occurred and so, in fact, the sequence HT cannot have occurred previously (other than on tosses $n-2$ and $n-1$). The only runs of tosses which don't contain an HT sequence are those of the form

$$\underbrace{TTT\cdots TT}_{k\text{ times}}\underbrace{HHH\ldots HH}_{l\text{ times}}$$

where $k, l \geqslant 0$.

We are looking for sequences of length n which end HTH and which, previously for $n-3$ tosses, have gone $TTT\cdots TTHHH\cdots HHH$. There are $n-2$ such sequences as k can take any integer value from 0 to $n-3$. Each list of Hs and Ts of length n is equally likely with probability 2^{-n}. So the probability we seek is

$$\sum_{n=3}^{\infty}\frac{n-2}{2^n}=\sum_{n=1}^{\infty}\frac{n}{2^{n+2}}=\frac{1}{8}\sum_{n=1}^{\infty}\frac{n}{2^{n-1}}.$$

Recalling that from the binomial theorem that

$$(1-x)^{-2}=\sum_{n=1}^{\infty}nx^{n-1}\qquad\text{for }\ |x|<1,$$

we see the desired probability is

$$\frac{1}{8}\left(1-\frac{1}{2}\right)^{-2}=\frac{4}{8}=\frac{1}{2}.$$