Solution (#432) Let $0 < a < b$. Two sequences are created by setting $a_1 = a$, $b_1 = b$ and recursively defining

$$a_{n+1} = \sqrt{a_n b_n}, \qquad b_{n+1} = \frac{a_n + b_n}{2}.$$

(i) Clearly $a_1 < b_1$ and if $a_1 < \cdots < a_n < b_n < \cdots < b_1$ then

$$\begin{aligned} a_{n+1} &= \sqrt{a_n b_n} > \sqrt{a_n a_n} = a_n; \\ b_{n+1} &= \frac{a_n + b_n}{2} < \frac{b_n + b_n}{2} = b_n. \end{aligned}$$

Finally $a_{n+1} < b_{n+1}$ from the AM-GM inequality.

(ii) We will show that $b_{n+1} - a_{n+1} \leqslant (b_n - a_n)/2$ which implies the required result. This inequality is equivalent to

$$a_n + b_n - 2\sqrt{a_n b_n} \leqslant b_n - a_n \quad \iff \quad 2a_n \leqslant 2\sqrt{a_n b_n} \quad \iff \quad a_n \leqslant b_n.$$

Hence the result follows and this means that a_n and b_n converge to the same limit, denoted $\operatorname{agm}(a, b)$.

(iii) If we begin with $\alpha_1 = 1$, $\beta_1 = b/a$ and define

$$\alpha_{n+1} = \sqrt{\alpha_n \beta_n}, \qquad \beta_{n+1} = \frac{\alpha_n + \beta_n}{2}.$$

We see now that $\alpha_n = a_n/a$ and $\beta_n = b_n/a$. This is true for $n = 1$ and if true at n then

$$\begin{aligned} \alpha_{n+1} &= \sqrt{\alpha_n \beta_n} = \sqrt{(a_n/a)(b_n/a)} = \sqrt{a_n b_n}/a = a_{n+1}/a; \\ \beta_{n+1} &= \frac{\alpha_n + \beta_n}{2} = \frac{a_n/a + b_n/a}{2} = \frac{1}{a}\left(\frac{a_n + b_n}{2}\right) = b_{n+1}/a. \end{aligned}$$

Hence $\alpha_n = a_n/a$ and so $\operatorname{agm}(1, b/a) = \operatorname{agm}(a, b)/a$ as required.

(iv) If we set $a_1 = 1$ and $b_1 = 1$ we get the sequences:

n	a_n	b_n
1	1	2
2	1.414213...	1.5
3	1.456475...	1.457106...
4	1.456791...	1.456791...

Hence $\operatorname{agm}(1, 2) = 1.45679$ to 5 decimal places.