Solution (#435) Let $0 < m < h$ and let $g(h,m)$ be as defined in #434. Recall that these values satisfy:

- $g(h,m) = g(h,m+1) + g(h-1,m-1) \qquad \text{for } 1 < m < h-1;$
- $g(h,h-1) = h-1$ for $h \geqslant 2$;
- $g(h,1) = g(h,2)$ for $h \geqslant 3$.

These three facts are sufficient to determine recursively all values of $g(h,m)$ where $0 < h < m$ as we saw in #434 – the second property defines $g(h,m)$ along the top diagonal of the grid, the third property provides us with the lowest value of the next diagonal, and then the first property gives us a recursion taking us along the next diagonal. For $0 < m < h$ we define

$$f(h,m) = \binom{2h-m-2}{h-m} - \binom{2h-m-2}{h-m-2},$$

and will show that $f(h,m)$ satisfies the same three properties, from which it follows that $g = f$.

Firstly for $h \geqslant 2$ we have

$$f(h,h-1) = \binom{h-1}{1} - \binom{h-1}{-1} = (h-1) - 0 = h-1.$$

We also note for $h \geqslant 3$ that

$$\begin{aligned} f(h,2) &= \binom{2h-4}{h-2} - \binom{2h-4}{h-4} = \frac{(2h-4)!}{(h-2)!(h-2)!} - \frac{(2h-4)!}{(h-4)!h!} \\ &= \frac{(2h-4)!}{(h-2)!h!}\{h(h-1) - (h-2)(h-3)\} \\ &= \frac{(2h-4)!}{(h-2)!h!}\{4h-6\} \\ &= \frac{(2h-3)!}{(h-2)!h!}\{2\} \\ &= \frac{(2h-3)!}{(h-2)!h!}\{h-(h-2)\} \\ &= \frac{(2h-3)!}{(h-1)!(h-2)!} - \frac{(2h-3)!}{(h-3)!h!} \\ &= \binom{2h-3}{h-1} - \binom{2h-3}{h-3} = f(h,1). \end{aligned}$$

Finally for $1 < m < h-1$ we have

$$\begin{aligned} f(h,m+1) + f(h-1,m-1) &= \left\{\binom{2h-m-3}{h-m-1} - \binom{2h-m-3}{h-m-3}\right\} + \left\{\binom{2h-m-3}{h-m} - \binom{2h-m-3}{h-m-2}\right\} \\ &= \left\{\binom{2h-m-3}{h-m-1} + \binom{2h-m-3}{h-m}\right\} - \left\{\binom{2h-m-3}{h-m-3} + \binom{2h-m-3}{h-m-2}\right\} \\ &= \binom{2h-m-2}{h-m} - \binom{2h-m-2}{h-m-2} \qquad \text{[by Lemma 2.17]} \\ &= f(h,m). \end{aligned}$$

As f satisfies the three defining properties of g then $f = g$.