Solution (#437) (i) The no-letters (or empty) word has even length. If U and V are even length Oxwords then the two words – aUb and UV – which may be produced by rules II and III also have even length. Hence the three rules can only generate even length *Oxwords.*

(ii) The only Oxword of length two is ab. The length four Oxwords are $aabb$ (rule II) and $abab$ (rule III). Finally the length six Oxwords are

$$aaabbb, \qquad aababb, \qquad ababab, \qquad abaabb, \qquad aabbab,$$

where the first two are produced with rule II and the last three via rule III.

(iii) The no-letters word has an equal number of as and bs. Again if this fact is true of Oxwords U and V then it is also true of the Oxwords aUb and UV which may be produced by rules II and III. Hence the three rules can only generate *Oxwords with equal numbers of* as and bs.

(iv) Let's say that a word W involving just the letters a and b has property C if it is even length, uses equal numbers of as and bs and is such that, for any i, there are at least as many as as bs amongst the first i letters of W. It is clear that rules II and III can only be used to produce words with property C. It remains to show that all words with property C can be produced using the three given rules.

Suppose, as our inductive hypothesis, that every word of length less than $2n$ with property C is an Oxword. This is true for $n = 1$ as ab is the only two-letter word with property C. Let W be a word of length $2n$ with property C. Such a word must necessarily start with an a; it must also end in a b or else there would be more bs than as in the first $2n - 1$ letters. Let k be the first occasion on which the first $2k$ letters of W include an equal number of as and bs. This certainly happens at least once as the first $2n$ letters (i.e. the whole word) have equal numbers of as and bs.

If $k = n$ is the first occasion then $W = a\tilde{W}b$ where $\tilde{W}$ is a word of length $2n - 2$ with property C. By hypothesis $\tilde{W}$ is an Oxword and so W is an Oxword by rule II.

If this happens at some earlier stage $k < n$, then the first $2k$ letters (call these U) have property C and the last $2n - 2k$ letters (call these V) also have property C. By hypothesis U and V are Oxwords and hence, by rule III, W is an Oxword.

(v) Let W be an Oxword of positive length. In part (iv) we noted that there is a first occasion when the first $2k$ letters of W form some Oxword U (where $k \geqslant 1$). We can write $U = aXb$ where X is an Oxword. (If X didn't have property C then there would be an earlier occasion of an Oxword than U.) So we may write $W = aXbY$ where X and Y are Oxwords. (Note that if $k = n$ then Y has no letters.)

Conversely suppose that $W = aXbY$ where W, X, Y are Oxwords. As X has property C then the first occasion on which the first $2k$ letters of W include equal numbers of as and bs is at aXb. Hence we see this is the only way to write an Oxword in this form.

(vi) Any Oxword of length $2n+2$ can be written uniquely as $aXbY$ where X is an Oxword of length $2i$ (for some i in the range $0 \leqslant i \leqslant n$) and Y is an Oxword of length $2n-2i$. Conversely any such expression $aXbY$ gives an Oxword of length $2n + 2$. Hence

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$