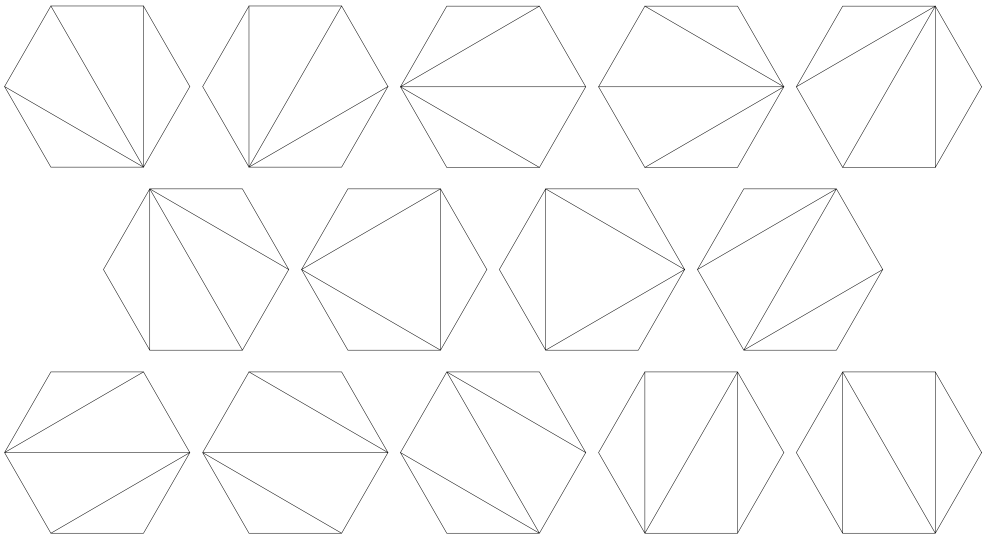
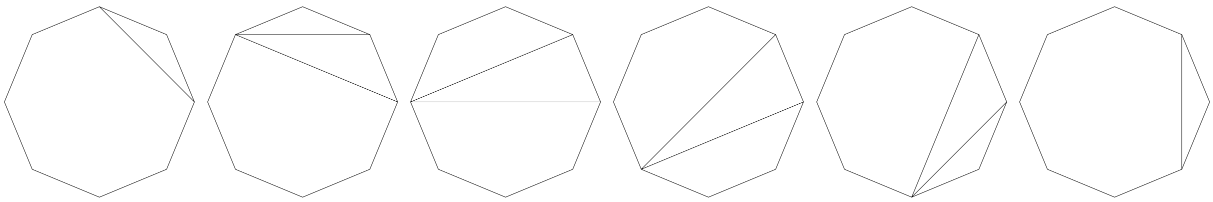
Solution (#438) The 14 ways to decompose a hexagon are drawn below.



Let C_n denote the number of ways of decomposing a polygon with $n+2$ sides into non-overlapping triangles whose vertices are vertices of the polygon. Consider such a polygon and choose one of its sides. This chosen side is part of a triangle whose third vertex is at one of the other n vertices. For example when $n = 6$ we have the possible arrangements below.



In the first diagram above we see that we have to decompose the remaining heptagon – there are C_5 ways of doing this. For the second diagram we have to decompose the remaining hexagon – there are C_4 ways of doing this. For the third diagram we have to decompose a quadrilateral and a pentagon. As we can do them independently then there are C_2C_3 ways of doing this. The remaining three diagrams are similar to the first three. In this case we have shown

$$C_6 = C_5 + C_4 + C_2C_3 + C_3C_2 + C_4 + C_5.$$

We have $C_1 = 1$ and might define C_0 as 1 so that the above reads as

$$C_6 = C_0C_5 + C_1C_4 + C_2C_3 + C_3C_2 + C_4C_1 + C_5C_0$$

which is the same recursion we found for the Catalan numbers in (2.27).

More generally consider a polygon with $n+2$ edges and fix a side. In any decomposition of the polygon this fixed side must be part of a triangle whose third vertex is at one of n other vertices. Counting in turn these n vertices, say the third vertex is at the kth vertex. If $k = 1$ or $k = n-1$ then (like the first and last diagrams above) there is a polygon with $n+1$ vertices that remains to decompose – there are C_{n-1} ways of doing this. If $2 \leqslant k \leqslant n-2$ then our triangle has split the original polygon into two polygons with $k+1$ and $n-k+2$ sides that remain to decompose. (These two numbers add to $n+3$; this is the original number of edges, subtracting one for the loss of the original chosen edge and adding two for the inclusion of the triangle's other edges.) We may decompose these two new polygons independently and there are $C_{k-1}C_{n-k}$ ways of doing this. Hence we have arrived at the recursion

$$\begin{aligned} C_n &= C_{n-1} + C_1C_{n-2} + C_2C_{n-3} + \cdots + C_{n-3}C_2 = C_{n-2}C_1 + C_{n-1} \\ &= C_0C_{n-1} + C_1C_{n-2} + C_2C_{n-3} + \cdots + C_{n-3}C_2 = C_{n-2}C_1 + C_{n-1}C_0 \end{aligned}$$

when we define $C_0 = 1$. As this is the same recursion as in (2.27) and $C_0 = C_1 = 1$ then the above C_n are indeed the Catalan numbers.