**Solution** (#448) Let $\alpha > 0$. If $\alpha = [q_0, q_1, \ldots, q_n]$ for some integers $q_0, q_1, \ldots, q_n$ then $\alpha$ is clearly rational.

Conversely suppose that $\alpha = a/b$ is a rational in its lowest form. We create here two sequences $q_i$ and $r_i$ of non-negative integers such that

$$\begin{aligned} a &= q_0 b + r_0, && \text{where } 0 \leqslant r_0 < b; \\ b &= q_1 r_0 + r_1, && \text{where } 0 \leqslant r_1 < r_0; \\ r_0 &= q_2 r_1 + r_2, && \text{where } 0 \leqslant r_2 < r_1, \end{aligned}$$

and so on. As the integer sequence is decreasing and bounded below by 0 then this process must eventually terminate with some $r_n = 0$.

Then

$$\begin{aligned} \lfloor \alpha \rfloor &= \left\lfloor \frac{a}{b} \right\rfloor = \left\lfloor q_0 + \frac{r_0}{b} \right\rfloor = q_0; \\ \alpha_1 &= \frac{1}{\alpha - \lfloor \alpha \rfloor} = \frac{1}{r_0/b} = \frac{b}{r_0} \qquad \text{so that } \lfloor \alpha_1 \rfloor = \left\lfloor \frac{b}{r_0} \right\rfloor = q_1 \text{ as } \frac{r_1}{r_0} < 1; \\ \alpha_2 &= \frac{1}{\alpha_1 - \lfloor \alpha_1 \rfloor} = \frac{1}{r_1/r_0} = \frac{r_0}{r_1} \qquad \text{so that } \lfloor \alpha_2 \rfloor = \left\lfloor \frac{r_0}{r_1} \right\rfloor = q_2 \text{ as } \frac{r_2}{r_1} < 1. \end{aligned}$$

We have from #442 that

$$\alpha = [\lfloor \alpha \rfloor, \alpha_1] = [\lfloor \alpha \rfloor, \lfloor \alpha_1 \rfloor, \alpha_2] = \cdots = [q_0, q_1] = [q_0, q_1, q_2] = \cdots$$

and know that this process eventually terminates with $\alpha_n$ being an integer and

$$\alpha = [\lfloor \alpha \rfloor, \lfloor \alpha_1 \rfloor, \lfloor \alpha_2 \rfloor, \ldots, \lfloor \alpha_{n-1} \rfloor, \alpha_n] = [q_0, q_1, q_2, \ldots, q_{n-1}, q_n].$$

**Remark:** Some may recognize that we have described the Euclidean Algorithm above to generate the sequences $q_i$ and $r_i$. The last non-zero remainder $r_{n-1}$ is the highest common factor of $a$ and $b$, the calculation of which is the purpose of the algorithm.