**Solution** (#462) (i) Let $d = 10$. Then $\sqrt{10} = [3, 6, 6, 6, \ldots]$ as $\alpha = \sqrt{10}$ and

$$\alpha_1 = \frac{1}{\sqrt{10} - 3} = \sqrt{10} + 3; \qquad \alpha_2 = \frac{1}{\left(\sqrt{10} + 3\right) - 6} = \sqrt{10} + 3, \ldots.$$

As the period of this continued fraction is 1 (and so odd), by Theorem 2.45 the fundamental solution is $\left(a_{2\times 1-1}, b_{2\times 1-1}\right)$. Now

$$\frac{a_1}{b_1} = 3 + \frac{1}{6} = \frac{19}{6}$$

and the fundamental solution is $X = 19$, $Y = 6$.

(ii) Let $d = 20$. Then $\sqrt{20} - [4, 2, 8, 2, 8, 2, 8, \ldots]$ as $\alpha = \sqrt{20}$ and

$$\alpha_1 = \left(\sqrt{20} - 4\right)^{-1} = \frac{\sqrt{5} + 2}{2}; \quad \alpha_2 = \left(\frac{\sqrt{5} + 2}{2} - 2\right)^{-1} = 2\left(\sqrt{5} + 2\right); \quad \alpha_3 = \left(2\sqrt{5} + 4 - 8\right)^{-1} = \frac{\sqrt{5} + 2}{2} = \alpha_1.$$

As the period of this continued fraction is 2 (and so even), by Theorem 2.45 the fundamental solution is $\left(a_{2-1}, b_{2-1}\right)$. Now

$$\frac{a_1}{b_1} = 4 + \frac{1}{2} = \frac{9}{2}$$

and the fundamental solution is $X = 9$, $Y = 2$.

(iii) Let $d = 41$. Then $\sqrt{41} - [6, 2, 2, 12, 2, 2, 12, 2, 2, \ldots]$ as $\alpha = \sqrt{41}$ and

$$\begin{aligned}
\alpha_1 &= \left(\sqrt{41} - 6\right)^{-1} = \frac{\sqrt{41} + 6}{5}; \qquad \alpha_2 = \left(\frac{\sqrt{41} + 6}{5} - 2\right)^{-1} = \frac{\sqrt{41} + 4}{5}; \\
\alpha_3 &= \left(\frac{\sqrt{41} + 4}{5} - 2\right)^{-1} = \sqrt{41} + 6; \qquad \alpha_4 = \left(\sqrt{41} + 6 - 12\right)^{-1} = \frac{\sqrt{41} + 6}{5} = \alpha_1.
\end{aligned}$$

As the period of this continued fraction is 3 (and so odd), by Theorem 2.45 the fundamental solution is $\left(a_{2\times 3-1}, b_{2\times 3-1}\right)$. Now

$$\begin{aligned}
\frac{a_5}{b_5} &= [6, 2, 2, 12, 2, 2] = [6, 2, 2, 12, 5/2] = [6, 2, 2, 62/5] \\
&= [6, 2, 129/62] = [6, 320/129] = 2049/320,
\end{aligned}$$

and the fundamental solution is $X = 2049$, $Y = 320$.

(iv) Let $d = 55$. Then $\sqrt{55} = [7, 2, 2, 2, 14, 2, 2, 2, 14, \ldots]$ as $\alpha = \sqrt{55}$ and

$$\begin{aligned}
\alpha_1 &= \left(\sqrt{55} - 7\right)^{-1} = \frac{\sqrt{55} + 7}{6}; \quad \alpha_2 = \left(\frac{\sqrt{55} + 7}{6} - 2\right)^{-1} = \frac{\sqrt{55} + 5}{5}; \quad \alpha_3 = \left(\frac{\sqrt{55} + 5}{5} - 2\right)^{-1} = \frac{\sqrt{55} + 5}{6}; \\
\alpha_4 &= \left(\frac{\sqrt{55} + 5}{6} - 2\right)^{-1} = \sqrt{55} + 7; \qquad \alpha_5 = \left(\sqrt{55} + 7 - 14\right)^{-1} = \frac{\sqrt{55} + 7}{6} = \alpha_1.
\end{aligned}$$

As the period of this continued fraction is 4 (and so even), by Theorem 2.45 the fundamental solution is $\left(a_{4-1}, b_{4-1}\right)$. Now

$$\frac{a_3}{b_3} = [7, 2, 2, 2] = [7, 2, 5/2] = [7, 12/5] = 89/12,$$

and the fundamental solution is $X = 89$, $Y = 12$.