Solution (#463) Let $d = 61$. Then we shall see that

$$\sqrt{61} = [7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, \ldots].$$

We have $\alpha = \sqrt{61}$ and then $\lfloor \alpha \rfloor = 7$ and

$$\alpha_1 = \frac{1}{\sqrt{61}-7} = \frac{\sqrt{61}+7}{12} \approx 1.23. \quad \alpha_2 = \left(\frac{\sqrt{61}+7}{12} - 1\right)^{-1} = \frac{12}{\sqrt{61}-5} = \frac{\sqrt{61}+5}{3} \approx 4.27.$$

$$\alpha_3 = \left(\frac{\sqrt{61}+5}{3} - 4\right)^{-1} = \frac{3}{\sqrt{61}-7} = \frac{\sqrt{61}+7}{4} \approx 3.70. \quad \alpha_4 = \left(\frac{\sqrt{61}+7}{4} - 3\right)^{-1} = \frac{4}{\sqrt{61}-5} = \frac{\sqrt{61}+5}{9} \approx 1.42.$$

$$\alpha_5 = \left(\frac{\sqrt{61}+5}{9} - 1\right)^{-1} = \frac{9}{\sqrt{61}-4} = \frac{\sqrt{61}+4}{5} \approx 2.36. \quad \alpha_6 = \left(\frac{\sqrt{61}+4}{5} - 2\right)^{-1} = \frac{5}{\sqrt{61}-6} = \frac{\sqrt{61}+6}{5} \approx 2.76.$$

$$\alpha_7 = \left(\frac{\sqrt{61}+6}{5} - 2\right)^{-1} = \frac{5}{\sqrt{61}-4} = \frac{\sqrt{61}+4}{9} \approx 1.31. \quad \alpha_8 = \left(\frac{\sqrt{61}+4}{9} - 1\right)^{-1} = \frac{9}{\sqrt{61}-5} = \frac{\sqrt{61}+5}{4} \approx 3.20.$$

$$\alpha_9 = \left(\frac{\sqrt{61}+5}{4} - 3\right)^{-1} = \frac{4}{\sqrt{61}-7} = \frac{\sqrt{61}+7}{3} \approx 4.93. \quad \alpha_{10} = \left(\frac{\sqrt{61}+7}{3} - 4\right)^{-1} = \frac{3}{\sqrt{61}-5} = \frac{\sqrt{61}+5}{12} \approx 1.06.$$

$$\alpha_{11} = \left(\frac{\sqrt{61}+5}{12} - 1\right)^{-1} = \frac{12}{\sqrt{61}-7} = \sqrt{61}+7 \approx 14.81. \quad \alpha_{12} = \left(\sqrt{61}+7-14\right)^{-1} = \frac{1}{\sqrt{61}-7} = \alpha_1.$$

As the continued fraction's period is 11, and so odd, then the fundamental solution is (a_{21}, b_{21}). We then have

$$\begin{aligned}
a_{21}/b_{21} &= [7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1] \\
&= [7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 1, 3, 5] \\
&= [7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 1, 16/5] \\
&= [7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 21/16] \\
&= [7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 58/21] \\
&= [7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 137/58] \\
&= [7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 195/137] \\
&= [7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 722/195] \\
&= [7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 3083/722] \\
&= [7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 3805/3083] \\
&= [7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 56353/3805] \\
&= [7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 60158/56353] \\
&= [7, 1, 4, 3, 1, 2, 2, 1, 3, 296985/60158] \\
&= [7, 1, 4, 3, 1, 2, 2, 1, 951113/296985] \\
&= [7, 1, 4, 3, 1, 2, 2, 1248098/951113] \\
&= [7, 1, 4, 3, 1, 2, 3447309/1248098] \\
&= [7, 1, 4, 3, 1, 8142716/3447309] \\
&= [7, 1, 4, 3, 11590025/8142716] \\
&= [7, 1, 4, 42912791/11590025] \\
&= [7, 1, 183241189/42912791] \\
&= [7, 226153980/183241189] \\
&= 1766319049/226153980,
\end{aligned}$$

so that the fundamental solution is $X = 1766319049$, $Y = 226153980$.