Solution (#466) The equation $6xy+1=x^2+y^2$ can be rewritten as

$$(x+y)^2-2(x-y)^2=-1.$$

So if (a,b) is a solution of $a^2-2b^2=-1$ we see that

$$x=\frac{a+b}{2}, \qquad y=\frac{a-b}{2},$$

solves $6xy+1=x^2+y^2$.

In #464 we found all integer solutions (a,b) of $a^2-2b^2=-1$. Further we related them to the solutions (c,d) of $c^2-2d^2=1$ by

$$a=c+2d, \qquad b=c+d.$$

And in #454 we showed that such c must be odd and such d must be even. Hence a and b are both odd and therefore x and y above are both integers.

Hence with the results of #464 we have integer solutions

$$x=\pm\left\{\frac{\left(1+\sqrt{2}\right)^{2n+4}-\left(1-\sqrt{2}\right)^{2n+4}}{4\sqrt{2}}\right\}, \qquad y=\pm\left\{\frac{\left(1+\sqrt{2}\right)^{2n+2}-\left(1-\sqrt{2}\right)^{2n+2}}{4\sqrt{2}}\right\},$$

of $6xy+1=x^2+y^2$.

On the other hand the equation $6xy=x^2+y^2+1$ rearranges to

$$(x+y)^2-2(x-y)^2=1.$$

If x and y are integers then so are $x\pm y$. But by #454 we then have $x+y$ is odd and $x-y$ is even. This is not possible and so the equation has no integer solutions.