Solution (#474) (i) Certainly $T_0(\cos\theta) = 1 = \cos(0\theta)$ and $T_1(\cos\theta) = \cos\theta = \cos(1\theta)$. If similar results hold for T_n and T_{n-1} then

$$\begin{aligned} T_{n+1}(\cos\theta) &= 2\cos\theta T_n(\cos\theta) - T_{n-1}(\cos\theta) \\ &= 2\cos\theta\cos n\theta - \cos(n-1)\theta \\ &= 2\cos\theta\cos n\theta - \{\cos n\theta\cos\theta + \sin n\theta\sin\theta\} \\ &= \cos\theta\cos n\theta - \sin n\theta\sin\theta \\ &= \cos(n+1)\theta, \end{aligned}$$

and the result follows by induction.

Likewise $U_0(\cos\theta)\sin\theta = \sin(1\theta)$ and $U_1(\cos\theta)\sin\theta = 2\cos\theta\sin\theta = \sin(2\theta)$. If similar results hold for U_n and U_{n-1} then

$$\begin{aligned} U_{n+1}(\cos\theta)\sin\theta &= \{2\cos\theta U_n(\cos\theta) - U_{n-1}(\cos\theta)\}\sin\theta \\ &= 2\cos\theta\sin(n+1)\theta - \sin n\theta \\ &= 2\cos\theta(\sin n\theta\cos\theta + \cos n\theta\sin\theta) - \sin n\theta \\ &= \sin n\theta(2\cos^2\theta - 1) + \cos n\theta(2\cos\theta\sin\theta) \\ &= \sin n\theta\cos 2\theta + \cos n\theta\sin 2\theta \\ &= \sin(n+2)\theta. \end{aligned}$$

The result follows by induction.

It follows that T_n has roots

$$\cos\left(\frac{k\pi}{2n}\right) \qquad \text{for } k = 1, 3, 5, \ldots, 2n-1.$$

These are n distinct real roots between -1 and 1. As T_n is a polynomial of degree n then these are all of T_n's roots. Similarly U_n has roots

$$\cos\left(\frac{k\pi}{n+1}\right) \qquad \text{for } k = 1, 2, 3, \ldots, n.$$

These are n distinct real roots between -1 and 1. As U_n is a polynomial of degree n then these are all of U_n's roots.

(ii) Note that

$$T_n(T_m(\cos\theta)) = T_n(\cos m\theta) = \cos nm\theta = T_{nm}(\cos\theta).$$

This means that $T_n(T_m(x)) = T_{nm}(x)$ for $-1 \leqslant x \leqslant 1$. As distinct polynomials cannot agree at more than a finite number of points it follows that

$$T_n(T_m(x)) = T_{nm}(x) \qquad \text{for all real } x.$$

(iii) Let $m \geqslant n$. Note that

$$\begin{aligned} 2T_m(\cos\theta)T_n(\cos\theta) &= 2\cos m\theta\cos n\theta \\ &= \cos(m+n)\theta + \cos(m-n)\theta \\ &= T_{m+n}(\cos\theta) + T_{m-n}(\cos\theta). \end{aligned}$$

Arguing as in part (ii) it follows that It follows that

$$2T_m(x)T_n(x) = T_{m+n}(x) + T_{m-n}(x) \qquad \text{for all real } x.$$