Solution (#479) (i) by considering the various ways P can reach a given point at time t we see that

- X_2 is distributed with probabilities $\frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{2}{9}, \frac{1}{9}$ on $-2, -1, 0, 1, 2$.
- X_3 is distributed with probabilities $\frac{1}{27}, \frac{3}{27}, \frac{6}{27}, \frac{7}{27}, \frac{6}{27}, \frac{3}{27}, \frac{1}{27}$ on $-3, -2, -1, 0, 1, 2, 3$.
- X_4 is distributed with probabilities $\frac{1}{81}, \frac{4}{81}, \frac{10}{81}, \frac{16}{81}, \frac{19}{81}, \frac{16}{81}, \frac{10}{81}, \frac{4}{81}, \frac{1}{81}$ on $-4, -3, -2, -1, 0, 1, 2, 3, 4$.

(ii) In a similar fashion if we consider the number of ways x^k can be attained when expanding the t brackets

$$(x^{-1}+1+x)(x^{-1}+1+x)\cdots(x^{-1}+1+x)$$

we see that there is an x^k term for each possible journey of P that ends at k. Each such journey has probability 3^{-t} and so the result follows. Note that these probabilities are the trinomial coefficients divided by 3^t (see #341).

(iii) By symmetry we can see that the mean of X_t is 0; for every path leading to k there is a corresponding one leading to $-k$ where all the $+1$ moves have been replaced by -1 moves and vice-versa. Alternatively we can write

$$3^{-t}(x^{-1}+1+x)^t = \sum_{k=-t}^{t} p_t^k x^k.$$

We are seeking to determine $\sum kp_t^k$. If we differentiate we obtain

$$3^{-t}t\left(1-x^{-2}\right)\left(x^{-1}+1+x\right)^{t-1} = \sum_{k=-t}^{t} kp_t^k x^{k-1}.$$

Setting $x=1$ we see that the mean is indeed 0. Arguing this way we can also work out the mean of $(X_t)^2$. We have

$$3^{-t}t\left(x-x^{-1}\right)\left(x^{-1}+1+x\right)^{t-1} = \sum_{k=-t}^{t} kp_t^k x^k.$$

Differentiating we have

$$3^{-t}t(t-1)\left(x-x^{-1}\right)(1-x^{-2})(x^{-1}+1+x)^{t-2} + 3^{-t}t\left(1+x^{-2}\right)\left(x^{-1}+1+x\right)^{t-1} = \sum_{k=-t}^{t} k^2 p_t^k x^{k-1}.$$

Setting $x=1$ again we have that

$$\sum_{k=-t}^{t} k^2 p_t^k = \frac{2t}{3}.$$

(This ties in with our previous result for the binomial random walk, where $(X_t)^2$ has mean t. For this trinomial walk only two-thirds of the time will P actually move.)