Solution (#482) (i) Without any loss of generality, say P initially moves vertically – this means that the following turn P moves horizontally and the time after that vertically. So on odd turns P is moves vertically and on even turns P moves horizontally. For P to be back at $(0,0)$ as many left as right moves must have been made, and as many up as down moves. So both these numbers must be even and given horizontal and vertical moves alternate the total number of moves must be a multiple of 4. (After $4n+2$ moves there will have been $2n+1$ horizontal moves which is not even.)

So the probability of being at $(0,0)$ is zero unless t is a multiple of 4.

(ii) Say $t = 4n$, after which times there will have been n moves in each direction. Of the $2n$ vertical moves n must be up and n down. The probability of this is

$$\frac{1}{2^{2n}}\binom{2n}{n}.$$

There is a similar probability of being back at the original horizontal. So the probability of being at $(0,0)$ equals

$$\frac{1}{2^{4n}}\binom{2n}{n}^2.$$

(iii) If $t = 2n$ then there will have been n moves that were horizontal. So $X_{2n} = N-(n-N) = 2N-n$ where N has the distribution $B(n,1/2)$. This means that X_{2n} can be $-n, 2-n, \ldots, n-2, n$ with the probability that $X_{2n} = 2i-n$

$$P(X_{2n} = 2i-n) = \frac{1}{2^n}\binom{n}{i} \quad \text{for } 0 \leqslant i \leqslant n.$$

If $t = 2n+1$ then there is a half chance of there having been n horizontal moves and a half chance of there having been $n+1$ horizontal moves. This means that X_{2n+1} can be $-n-1, -n, 1-n, \ldots, n-1, n, n+1$ with

$$\begin{aligned} P(X_{2n+1} &= 2i-n) = \frac{1}{2^{n+1}}\binom{n}{i} \quad \text{for } 0 \leqslant i \leqslant n; \\ P(X_{2n+1} &= 2i-n-1) = \frac{1}{2^{n+2}}\binom{n+1}{i} \quad \text{for } 0 \leqslant i \leqslant n+1; \end{aligned}$$

(iv) By symmetry we can see that both X_{2n} and X_{2n+1} have zero means.

(v) As $X_{2n} = 2N-n$ where N has distribution $B(n,1/2)$ then

$$(X_{2n})^2 = (2N-n)^2 = 4N^2 - 4Nn + n^2$$

has mean

$$4 \times \left(\frac{n}{4} + \frac{n^2}{4}\right) - 4 \times \left(\frac{n}{2}\right) \times n + n^2 = n + n^2 - 2n^2 + n^2 = n.$$

The mean value of $(X_{2n+1})^2$ equals

$$\frac{1}{2}(n) + \frac{1}{2}(n+1) = n + \frac{1}{2}.$$

So the mean of $(X_t)^2$ is $t/2$. Similarly the mean of $(Y_t)^2$ is $t/2$.

So the mean of $(X_t)^2 + (Y_t)^2$ equals t.