Solution (#485) Suppose that the sequence $a_1, a_2, a_3, \ldots$ is complete. As the sequence is increasing then a_1 is necessarily 1 or else it would be impossible to represent 1 as a sum. Suppose as an inductive hypothesis that

$$1 + a_1 + a_2 + \cdots + a_k \geqslant a_{k+1}$$

is true for each $1 \leqslant k < n$. If it were the case that

$$a_1 + a_2 + \cdots + a_n < a_{n+1} - 1,$$

then $a_{n+1} - 1$ would not be expressible as a sum of the a_i. Hence we have

$$1 + a_1 + a_2 + \cdots + a_n \geqslant a_{n+1}.$$

The result follows by induction.

Conversely say that $a_1 = 1$ and

$$1 + a_1 + a_2 + \cdots + a_k \geqslant a_{k+1} \qquad \text{for each } k \geqslant 1.$$

Then 1 is expressible as a sum. As an inductive hypothesis, suppose that all numbers from 1 to $a_1 + a_2 + \cdots + a_k$ inclusive are expressible as a sum involving the numbers $a_1, \ldots, a_k$. Say now that

$$a_1 + a_2 + \cdots + a_k < x \leqslant a_1 + a_2 + \cdots + a_k + a_{k+1}.$$

Then

$$1 \leqslant x - a_{k+1} \leqslant a_1 + a_2 + \cdots + a_k.$$

By hypothesis there is a selection of $a_1, \ldots, a_k$ that sums to $x - a_{k+1}$ and hence there is a selection of $a_1, \ldots, a_{k+1}$ that sums to x. The result follows by induction.