Solution (#487) (i) Zeckendorf's theorem shows that the Fibonacci sequence $F_1, F_2, F_3, F_4, \ldots$ is complete; moreso, in that a Zeckendorf representation uses non-consecutive Fibonacci numbers from $F_2, F_3, F_4, \ldots$. Note crucially that F_1 is never used in a Zeckendorf representation but is in the sequence we are aiming to show is complete even when missing a member.

Suppose now that F_N is removed from the sequence $F_1, F_2, F_3, F_4, \ldots$. If a positive integer n has a Zeckendorf representation that doesn't use F_N then we can continue to represent n in this way using the reduced list. Suppose instead that F_N was used in the representation. This means that F_{N-1} was not used (as a Zeckendorf representation involves non-consecutive Fibonacci numbers). If it was also the case that F_{N-2} was not used then we can replace F_N with F_{N-1} and F_{N-2} as

$$F_N = F_{N-1} + F_{N-2}.$$

If F_{N-2} was used then F_{N-3} was not and suppose F_{N-4} was not used; then we make the replacement

$$F_N = F_{N-1} + F_{N-3} + F_{N-4}.$$

If F_{N-2} and F_{N-4} were used then F_{N-3} and F_{N-5} were not and suppose F_{N-6} was not used; then we make the replacement

$$F_N = F_{N-1} + F_{N-3} + F_{N-5} + F_{N-6}.$$

We can continue this process working backwards through the Fibonacci numbers. If we come to a point where two consecutive Fibonacci numbers weren't used in the representation of n then the process stops there. If no such consecutive pair is arrived at, this means that $F_{N-2}, F_{N-4}, F_{N-6,\ldots}$ were used in the representation. In this case the process leads to the replacements

$$\begin{aligned} N \text{ even} &: \qquad F_N = F_{N-1} + F_{N-3} + F_{N-5} + \cdots + F_1; \\ N \text{ odd} &: \qquad F_N = F_{N-1} + F_{N-3} + F_{N-5} + \cdots + F_2 + F_1, \end{aligned}$$

both of which are valid replacements as F_1 is never used in the Zeckendorf representation of a number.

(ii) Let the increasing sequence $b_1, b_2, b_3, \ldots$ have the property that, even when some element is removed, the sequence remains complete. We aim to show that $b_n \leqslant F_n$ for all $n \geqslant 1$.

Suppose for a contradiction that this is not the case and let b_N be the first occasion in which $b_N > F_N$. Recall the identity

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$$

from #347. So

$$b_1 + b_2 + \cdots + b_{N-2} \leqslant F_1 + F_2 + \cdots + F_{N-2} = F_N - 1.$$

Thus if we removed b_{N-1} from the sequence we would not be able to produce F_N as a sum. This is the required contradiction.