Solution (#1056) Let

$$D = \begin{pmatrix} 5 & -3 & -5 \\ 2 & 9 & 4 \\ -1 & 0 & 7 \end{pmatrix},$$

and $T = \mu_D$. We are asked to find a basis $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}_3$ such that

$$_{\mathcal{V}}T_{\mathcal{V}} = \begin{pmatrix} a & 0 & 0 \\ 0 & c & e \\ 0 & 0 & f \end{pmatrix}.$$

As $c_D(x) = (x-6)^2(x-9)$ this means that a, c, f equal $6, 6, 9$ in some order. It also means that $\mathbf{v}_1$ is an a-eigenvector of T and that $\mathbf{v}_2$ is a c-eigenvector.

So we might choose $\mathbf{v}_1 = (-2, 1, 1)^T$ and $a = 9$, and choose $\mathbf{v}_2 = (1, -2, 1)^T$ and $c = 6$. Given those choices it means that $f = 6$.

We then need a third vector $\mathbf{v}_3$, independent of $\mathbf{v}_1$ and $\mathbf{v}_2$, and such that $D\mathbf{v}_3 = e\mathbf{v}_2 + 6\mathbf{v}_3$ or equivalently

$$(D - 6I)\mathbf{v}_3 = e\mathbf{v}_2.$$

If such a vector $\mathbf{v}_3$ exists then a non-zero multiple of it will also have the desired properties so we can assume $e = 1$ without any loss of generality. So we have the system

$$\left(\begin{array}{ccc|c} -1 & -3 & -5 & 1 \\ 2 & 3 & 4 & -2 \\ -1 & 0 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 5 & -1 \\ 0 & -3 & -6 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

which has solution $\mathbf{v}_3 = (0, -2, 1)^T$ and

$$\mathcal{V} = \left\{(-2, 1, 1)^T, (1, -2, 1)^T, (0, -2, 1)^T\right\}; \qquad {}_{\mathcal{V}}T_{\mathcal{V}} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 6 \end{pmatrix}.$$

Taking powers of $_{\mathcal{V}}T_{\mathcal{V}}$ it is fairly clear we will have

$$_{\mathcal{V}}(T^n)_{\mathcal{V}} = \begin{pmatrix} 9^n & 0 & 0 \\ 0 & 6^n & c_n \\ 0 & 0 & 6^n \end{pmatrix}$$

and then the c_n must satisfy the recurrence relation

$$c_{n+1} = 6c_n + 6^n, \qquad c_1 = 1.$$

The solution of this recurrence is $c_n = n6^{n-1}$.

Now

$$D^n = {}_{\mathcal{E}}(T^n)_{\mathcal{E}} = ({}_{\mathcal{E}}I_{\mathcal{V}})\,({}_{\mathcal{V}}(T^n)_{\mathcal{V}})\,({}_{\mathcal{V}}I_{\mathcal{E}}).$$

We have

$$_{\mathcal{E}}I_{\mathcal{V}} = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix}, \qquad {}_{\mathcal{V}}I_{\mathcal{E}} = ({}_{\mathcal{E}}I_{\mathcal{V}})^{-1} = \frac{1}{3}\begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 4 \\ -3 & -3 & -3 \end{pmatrix}.$$

Hence

$$\begin{aligned} D^n &= \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 9^n & 0 & 0 \\ 0 & 6^n & n6^{n-1} \\ 0 & 0 & 6^n \end{pmatrix} \frac{1}{3} \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 4 \\ -3 & -3 & -3 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} -2 \times 9^n & 6^n & n6^{n-1} \\ 9^n & -2 \times 6^n & -2(n+6)6^{n-1} \\ 9^n & 6^n & (n+6)6^{n-1} \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 4 \\ -3 & -3 & -3 \end{pmatrix} \\ &= \begin{pmatrix} (6-n)6^{n-1} & -6 \times 9^{n-1} + 4 \times 6^{n-1} - n6^{n-1} & -12 \times 9^{n-1} + 8 \times 6^{n-1} - n6^{n-1} \\ 2n6^{n-1} & 3 \times 9^{n-1} + (2n+4)6^{n-1} & 6 \times 9^{n-1} + (2n-4)6^{n-1} \\ -n6^{n-1} & 3 \times 9^{n-1} - (n+2)6^{n-1} & 6 \times 9^{n-1} + (2-n)6^{n-1} \end{pmatrix}. \end{aligned}$$