Solution (#1090) Let A be a 3×3 matrix such that $A(\mathbf{v} \wedge \mathbf{w}) = A\mathbf{v} \wedge A\mathbf{w}$ for all $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}_3$. Setting $\mathbf{v} = \mathbf{i}$, $\mathbf{w} = \mathbf{j}$ we have

$$A\mathbf{k} = A\mathbf{i} \wedge A\mathbf{j}$$

and then by #911

$$|A\mathbf{k}|^2 = A\mathbf{k} \cdot A\mathbf{k} = [A\mathbf{i}, A\mathbf{j}, A\mathbf{k}] = \det A \times [\mathbf{i}, \mathbf{j}, \mathbf{k}] = \det A.$$

Similarly we can show

$$|A\mathbf{i}|^2 = |A\mathbf{j}|^2 = |A\mathbf{k}|^2 = \det A$$

and in particular $\det A > 0$. We also have

$$A\mathbf{i} \cdot A\mathbf{k} = A\mathbf{i} \cdot (A\mathbf{i} \wedge A\mathbf{j}) = 0$$

and likewise

$$A\mathbf{i} \cdot A\mathbf{j} = A\mathbf{j} \cdot A\mathbf{k} = A\mathbf{i} \cdot A\mathbf{k} = 0.$$

This means that

$$\frac{A\mathbf{i}}{\sqrt{\det A}}, \quad \frac{A\mathbf{j}}{\sqrt{\det A}}, \quad \frac{A\mathbf{k}}{\sqrt{\det A}},$$

is an orthonormal basis and so $A = \sqrt{\det A}P$ where P is an orthogonal matrix with $\det P = 1$. Thus, by Corollary 3.157(b),

$$\det A = \det\left(\sqrt{\det A}P\right) = \left(\sqrt{\det A}\right)^3 \det P = (\det A)^{3/2}$$

and hence $\det A = 1$. Hence A is necessarily orthogonal with $\det A = 1$.