Solution (#1102) Note that $R(\mathbf{i},\theta)$ fixes $\mathbf{i}$ and $R(\mathbf{j},\theta)$ fixes $\mathbf{j}$. Hence

$$R(\mathbf{j},\beta)\,R(\mathbf{i},\gamma)\mathbf{i} = R(\mathbf{j},\beta)\mathbf{i} = \cos\beta\mathbf{i} + \sin\beta\mathbf{k}.$$

We need this to equal

$$R(\mathbf{i},-\alpha)B\mathbf{i} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \frac{1}{2} \begin{pmatrix} \sqrt{2} \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{2} \\ \cos\alpha + \sin\alpha \\ -\sin\alpha + \cos\alpha \end{pmatrix}.$$

Hence we need that $\cos\alpha + \sin\alpha = 0$ which occurs only at $\alpha = -\pi/4$ for the given range of α. Further we then have

$$\cos\beta = 1/\sqrt{2} \qquad \sin\beta = 1/\sqrt{2}.$$

For β in the given range this only occurs when $\beta = \pi/4$. Finally then we need $R(\mathbf{i},\gamma) = R(\mathbf{j},-\beta)R(\mathbf{i},-\alpha)B$. This product equals

$$\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \frac{1}{2} \begin{pmatrix} \sqrt{2} & \sqrt{2} & 0 \\ 1 & -1 & \sqrt{2} \\ 1 & -1 & -\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 2 \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = R(\mathbf{i},-\pi/2).$$

So $\alpha = -\pi/4$, $\beta = \pi/4$, $\gamma = -\pi/2$.