Solution (#1103) Let R denote an orthogonal 3×3 matrix with $\det R = 1$ and let $R(\mathbf{i},\theta)$ and $R(\mathbf{j},\theta)$ be as in #1101.

(i) Suppose that $R\mathbf{i} = \mathbf{i}$. Then the first column of R's matrix is $(1,0,0)^T$. Further as the first row of R is a unit vector then R's first row is $(1,0,0)$. So we have $R = \operatorname{diag}(1,Q)$ for some 2×2 matrix Q. As

$$I_3 = R^T R = \operatorname{diag}\left(1,Q^T\right)\operatorname{diag}(1,Q) = \operatorname{diag}\left(1,Q^T Q\right)$$

then $Q^T Q = I_2$ and so Q is orthogonal. Further as $\det R = 1$ then $\det Q = 1$. By Example 4.18 we see $Q = R_\theta$ for some θ in the range $-\pi < \theta \leqslant \pi$ and hence

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

as required.

(ii) In the absence of the condition $R\mathbf{i} = \mathbf{i}$, it still remains the case that $R\mathbf{i}$ is a unit vector as R is orthogonal. Say $R\mathbf{i} = (x,y,z)$ where $x^2+y^2+z^2 = 1$. We wish to find c,d,α such that $R(\mathbf{i},\alpha)^{-1}R\mathbf{i} = c\mathbf{i} + d\mathbf{k}$ or equivalently

$$\begin{pmatrix} x \\ y\cos\alpha + z\sin\alpha \\ -y\sin\alpha + z\cos\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ d \end{pmatrix}.$$

Hence we must set $c = x$. We further see that we need to choose α so that $\tan\alpha = -y/z$ and set $d = -y\sin\alpha + z\cos\alpha$. There are two choices of α in the range $-\pi < \alpha \leqslant \pi$ which differ by π. Hence the two different α lead to the same value of d save for its sign and we should choose the α that leads to $d > 0$. (The exception to this is when $R\mathbf{i} = \mathbf{i}$ already in which case any choice of α will do and we would have $d = 0$.)

We now need to determine β such that $R(\mathbf{j},\beta)^{-1}(c\mathbf{i} + d\mathbf{k}) = \mathbf{i}$. This is equivalent to

$$\begin{pmatrix} c\cos\beta + d\sin\beta \\ 0 \\ -c\sin\beta + d\cos\beta \end{pmatrix} = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}\begin{pmatrix} c \\ 0 \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{11.2}$$

As $c^2 + d^2 = 1$ and $d \geqslant 0$ we see that there is unique β in the range $0 \leqslant \beta \leqslant \pi$ such that $c = \cos\beta$ and $d = \sin\beta$. For this choice of β we see that (11.2) is true.

(iii) For these choices of α and β we have

$$R(\mathbf{j},\beta)^{-1}R(\mathbf{i},\alpha)^{-1}R\mathbf{i} = \mathbf{i}.$$

So $R(\mathbf{j},\beta)^{-1}R(\mathbf{i},\alpha)^{-1}R$ is an orthogonal, determinant 1 matrix which fixes $\mathbf{i}$. By (i) we know that

$$R(\mathbf{j},\beta)^{-1}R(\mathbf{i},\alpha)^{-1}R = R(\mathbf{i},\gamma)$$

for some γ in the range $-\pi < \gamma \leqslant \pi$ and the required result follows.