Solution (#1109) Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be an orthonormal set (and so independent). If we apply the Gram-Schmidt process to these vectors then at the first step we find

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|} = \frac{\mathbf{v}_1}{1} = \mathbf{v}_1.$$

Now say, as an inductive hypothesis, that $\mathbf{w}_i = \mathbf{v}_i$ for $i < I$ when the Gram-Schmidt process has been applied. Then

$$\begin{aligned}
\mathbf{y}_I &= \mathbf{v}_I - (\mathbf{v}_I \cdot \mathbf{w}_1)\,\mathbf{w}_1 - (\mathbf{v}_I \cdot \mathbf{w}_2)\,\mathbf{w}_2 - \cdots (\mathbf{v}_I \cdot \mathbf{w}_{I-1})\,\mathbf{w}_{I-1} \\
&= \mathbf{v}_I - (\mathbf{v}_I \cdot \mathbf{v}_1)\,\mathbf{v}_1 - (\mathbf{v}_I \cdot \mathbf{v}_2)\,\mathbf{v}_2 - \cdots (\mathbf{v}_I \cdot \mathbf{v}_{I-1})\,\mathbf{v}_{I-1} \qquad \text{[by hypothesis]} \\
&= \mathbf{v}_I
\end{aligned}$$

as the $\mathbf{v}_i$ are orthonormal. Hence

$$\mathbf{w}_I = \frac{\mathbf{y}_I}{|\mathbf{y}_I|} = \frac{\mathbf{v}_I}{1} = \mathbf{v}_I.$$

The result follows by induction.