Solution (#1130) Let

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}}\,(1,1)\,, \qquad \mathbf{e}_1 = \frac{1}{\sqrt{2}}\,(-1,1)\,.$$

Note that

$$\begin{aligned}
\mathbf{e}_1 \cdot \mathbf{e}_1 &= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1; \\
\mathbf{e}_2 \cdot \mathbf{e}_2 &= \left(\frac{-1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1; \\
\mathbf{e}_1 \cdot \mathbf{e}_2 &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{-1}{2} + \frac{1}{2} = 1,
\end{aligned}$$

showing that $\mathbf{e}_1, \mathbf{e}_2$ is an orthonormal basis for $\mathbb{R}_2$.

Say now that $X\mathbf{e}_1 + Y\mathbf{e}_2 = x\mathbf{i} + y\mathbf{j}$. Then

$$x = \frac{X-Y}{\sqrt{2}}; \qquad y = \frac{X+Y}{\sqrt{2}}.$$

So

$$\begin{aligned}
x^2 + xy + y^2 = 1 &\iff \left(\frac{X-Y}{\sqrt{2}}\right)^2 + \left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + \left(\frac{X+Y}{\sqrt{2}}\right)^2 = 1 \\
&\iff (X^2 - 2XY + Y^2) + \left(X^2 - Y^2\right) + \left(X^2 + 2XY + Y^2\right) = 2 \\
&\iff 3X^2 + Y^2 = 2 \\
&\iff \frac{X^2}{(\sqrt{2/3})^2} + \frac{Y^2}{(\sqrt{2})^2} = 1.
\end{aligned}$$

Hence the curve is an ellipse and its area is

$$\pi ab = \pi \times \sqrt{\frac{2}{3}} \times \sqrt{2} = \frac{2\pi}{\sqrt{3}}.$$