Solution (#1165) Let $\mathbf{v}$ and $\mathbf{w}$ be independent vectors in $\mathbb{R}^2$. By rotating our axes, if necessary, we can assume that $\mathbf{v} = (a, 0)$ where $a > 0$ and that $\mathbf{w} = (c, d)$; the independence of $\mathbf{v}$ and $\mathbf{w}$ means that $d \neq 0$. If we write $\mathbf{r}(t) = (x(t), y(t))$ then we have

$$x(t) = a\cos t + c\sin t, \qquad y(t) = d\sin t.$$

So

$$\cos t = \frac{dx - cy}{ad}; \qquad \sin t = \frac{y}{d}.$$

As $\cos^2 t + \sin^2 t = 1$ then we see that our conic has equation

$$\left(\frac{dx - cy}{ad}\right)^2 + \left(\frac{y}{d}\right)^2 = 1,$$

which rearranges to

$$d^2x^2 - 2cdxy + \left(c^2 + a^2\right) y^2 = a^2d^2.$$

Note – in the notation of Theorem 4.53 – that

$$B^2 - 4AC = 4c^2d^2 - 4d^2(a^2 + c^2) = -4d^2a^2 < 0,$$

so that the curve is an ellipse, single point or empty set. As it is clearly neither of the last two cases the curve is an ellipse.