Solution (#1170) (i) Let C denote the unit circle $x^2 + y^2 = 1$ and let $P = (X, Y)$ be a point outside the circle. As P is outside the circle then $X^2 + Y^2 > 1$ and let $Q = (a, b)$ be a point on C. By #1164 the tangent to C at Q has equation

$$(a, b, 1)\,\mathrm{diag}(1, 1, -1)(x, y, 1) = 0$$

which multiplies out as

$$ax + by = 1.$$

The point P lies on this line when

$$aX + bY = 1, \qquad a^2 + b^2 = 1.$$

As we can uniquely write $a = \cos\theta$ and $b = \sin\theta$ for some $0 \leqslant \theta < 2\pi$, then we are seeking to solve the equation

$$1 = X\cos\theta + Y\sin\theta = \sqrt{X^2 + Y^2}\cos(\theta - \alpha)$$

for some α. As $X^2 + Y^2 > 1$ then there are two values of θ which solve this equation.

So there are two tangent lines to C

$$a_1 x + b_1 y = 1, \qquad a_2 x + b_2 y = 1$$

which contain P and these lines are respectively tangent to C at $Q_1 = (a_1, b_1)$ and $Q_2 = (a_2, b_2)$. As (X, Y) lies on these lines we also have that

$$a_1 X + b_1 Y = 1, \qquad a_2 X + b_2 Y = 1$$

and so $xX + yY = 1$ is the equation of the line $Q_1 Q_2$.

In this way we see that any point (X, Y) outside the circle corresponds to a line $Xx + Yy = 1$ which intersects C twice. Any line that intersects C twice can be written in the form

$$\alpha x + \beta y = \gamma \qquad \text{where} \qquad \alpha^2 + \beta^2 > \gamma^2.$$

Except for diameters, when $\gamma = 0$, such lines can be put in the form $(\alpha/\gamma)x + (\beta/\gamma)y = 1$ which is the polar of $(\alpha/\gamma, \beta/\gamma)$ which lies outside the circle. The diameters can be viewed as the polars of "points at infinity" where parallel lines meet but are not the polars of points in the xy-plane.

(ii) Let l be a line $ax + by = 1$ which does not intersect the unit circle C, so that $a^2 + b^2 < 1$. Take a point $P = (X, Y)$ on l so that $aX + bY = 1$. Let l_P denote its polar which has equation $Xx + Yy = 1$. Note that the point (a, b) lies on the line l_P. As $P = (X, Y)$ is an arbitrary point of the line l then all the polars pass through (a, b). To every line $ax + by = 1$ with $a^2 + b^2 < 1$ which does not meet C we can associate a point (a, b) inside the circle C and vice versa. The only choice of (a, b) with $a^2 + b^2 < 1$ which does not correspond to a line $ax + by = 1$ is the centre where $a = b = 0$. The centre's polar can be viewed as the "line at infinity" which contains all the points at infinity mentioned above.

(iii) A point $P = (a, b)$, other than the origin, has polar l_P with equation $ax + by = 1$. The inverse point of P is the point

$$I = \lambda(a, b) \qquad \text{where } \lambda > 0$$

and such that

$$1 = |OI|\,|OP| = \lambda\sqrt{a^2 + b^2}\sqrt{a^2 + b^2} = \lambda(a^2 + b^2).$$

Hence

$$I = \frac{(a, b)}{a^2 + b^2}.$$

We see that I lies on l_P and further the line OP, with equation $by = ax$, and l_P are perpendicular as their gradients multiply to -1.