Solution (#1188) Assume that $\mathbf{a}$ and $\mathbf{b}$ are non-zero, as clearly the equation has no solutions if either is zero. We may choose co-ordinates so that

$$\mathbf{a} = (1,0,0) \qquad \text{and} \qquad \mathbf{b} = (b\cos\theta, b\sin\theta, 0),$$

where $0 \leqslant \theta \leqslant \pi$. Then

$$\begin{aligned} \mathbf{r}\wedge\mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ 1 & 0 & 0 \end{vmatrix} = (0, z, -y)\,; \\ \mathbf{r}\wedge\mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ b\cos\theta & b\sin\theta & 0 \end{vmatrix} = b\,(-z\sin\theta, -z\cos\theta, x\sin\theta - y\cos\theta)\,. \end{aligned}$$

So

$$(\mathbf{r}\wedge\mathbf{a})\cdot(\mathbf{r}\wedge\mathbf{b}) = b\left(-z^2\cos\theta - xy\sin\theta + y^2\cos\theta\right).$$

We can then rewrite $(\mathbf{r}\wedge\mathbf{a})\cdot(\mathbf{r}\wedge\mathbf{b}) = 1$ as

$$(x,y,z)\begin{pmatrix} 0 & -\sin\theta & 0 \\ -\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & -2\cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{2}{b}.$$

The characteristic polynomial of this matrix is

$$\begin{aligned} & \begin{vmatrix} x & \sin\theta & 0 \\ \sin\theta & x - 2\cos\theta & 0 \\ 0 & 0 & x + 2\cos\theta \end{vmatrix} \\ = {} & (x + 2\cos\theta)\left(x^2 - 2x\cos\theta - \sin^2\theta\right) \\ = {} & (x + 2\cos\theta)\,(x - 1 - \cos\theta)\,(x + 1 - \cos\theta)\,. \end{aligned}$$

By the spectral theorem we can make an orthonormal change of co-ordinates so that the given equation becomes

$$(-2\cos\theta)\,X^2 + (1+\cos\theta)Y^2 + (\cos\theta - 1)Z^2 = \frac{2}{b}. \tag{11.8}$$

We have several cases to consider (i) $\cos\theta = 0$, (ii) $\cos\theta = -1$, (iii) $\cos\theta = 1$, (iv) $0 < \cos\theta < 1$, (v) $-1 < \cos\theta < 0$. In cases (i), (ii), (iii) we have the equations

$$Y^2 - Z^2 = 2/b, \qquad 2X^2 - 2Z^2 = 2/b, \qquad -2X^2 + 2Y^2 = 2/b,$$

each of which is a hyperbolic cylinder. In case (iv) two of the coefficients in (11.8) are negative, one positive, and so we have a hyperboloid of two sheets. In case (v) two of the coefficients are positive and one is negative so that we have a hyperboloid of one sheet.

Thus the equation $(\mathbf{r}\wedge\mathbf{a})\cdot(\mathbf{r}\wedge\mathbf{b}) = 1$ defines a hyperboloid of two sheets when $0 < \cos\theta < 1$. As $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$ then this inequality is equivalent to

$$0 < \mathbf{a}\cdot\mathbf{b} < |\mathbf{a}|\,|\mathbf{b}|\,.$$