Solution (#1189) Let A and B be $n \times n$ symmetric square matrices with the same characteristic polynomial. By the spectral theorem the roots of the characteristic polynomial are all real. So we have

$$c_A(x) = c_B(x) = (x - \lambda_1)^{n_1}(x - \lambda_2)^{n_2} \cdots (x - \lambda_r)^{n_r}$$

where n_i is the (algebraic and geometric) multiplicity of λ_i. Again by the spectral theorem there are orthogonal matrices Q and R such that

$$Q^T A Q = \mathrm{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \ldots, \lambda_r I_{n_r}) = R^T B R.$$

Hence

$$(QR^T)^T A (QR^T) = RQ^T AQR^T = B$$

and we should take $P = QR^T$.

Say now that the eigenvalues are distinct. If we fix R then we see there are as many Q as there are P and so we can instead ask the question: how many orthogonal Q are there such that

$$Q^T A Q = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

We see then that the first column of Q is a λ_1-eigenvector. Further as Q is orthogonal then that column must also be a unit vector. As the λ_1-eigenspace is 1-dimensional then there are precisely two λ_1-eigenvectors of unit length. The same follows for each column of Q and so in all there are 2^n such Q, and the same number of such P.