Solution (#1204) Let M be a complex $n \times n$ matrix M such that $M = \overline{M}^T$.

(i) Firstly we show that the roots of $c_M(x)$ are real in a similar fashion to Proposition 4.54. Let λ be a root of $c_M(x)$. Then there is a non-zero complex vector $\mathbf{v}$ in $\mathbb{C}^n$ such that $M\mathbf{v} = \lambda\mathbf{v}$. Applying conjugation and the transpose to this equation we have

$$\overline{M}\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}} \quad \Longrightarrow \quad M^T\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}} \quad \Longrightarrow \quad \bar{\mathbf{v}}^T M = \bar{\lambda}\bar{\mathbf{v}}^T.$$

We then have

$$\bar{\lambda}\bar{\mathbf{v}}^T\mathbf{v} = \bar{\mathbf{v}}^T M\mathbf{v} = \bar{\mathbf{v}}^T\lambda\mathbf{v} = \lambda\bar{\mathbf{v}}^T\mathbf{v}.$$

Now for any non-zero *complex* vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ we have

$$\bar{\mathbf{v}}^T\mathbf{v} = \bar{\mathbf{v}} \cdot \mathbf{v} = \overline{v_1}v_1 + \cdots + \overline{v_n}v_n = |v_1|^2 + \cdots + |v_n|^2 > 0.$$

As $(\bar{\lambda} - \lambda)\bar{\mathbf{v}}^T\mathbf{v} = 0$ then $\lambda = \bar{\lambda}$ and so λ is real.

(ii) As argued in Proposition 4.54(b) it also follows that when $M\mathbf{v} = \lambda\mathbf{v}$ and $M\mathbf{w} = \mu\mathbf{w}$ where $\lambda \neq \mu$ are distinct reals, we have

$$\lambda\bar{\mathbf{v}}^T\mathbf{w} = (\overline{\lambda\mathbf{v}})^T\mathbf{w} = (\overline{M\mathbf{v}})^T\mathbf{w} = \bar{\mathbf{v}}^T\overline{M}^T\mathbf{w} = \bar{\mathbf{v}}^T M\mathbf{w} = \bar{\mathbf{v}}^T\mu\mathbf{w} = \mu\bar{\mathbf{v}}^T\mathbf{w}.$$

As $\lambda \neq \mu$ then $\bar{\mathbf{v}}^T\mathbf{w} = 0$.

(iii) As in the proof of the spectral theorem we shall prove the result by strong induction on n. When $n = 1$ there is nothing to prove as all 1×1 matrices are diagonal and so we can simply take $U = I_1$.

Suppose now that the result holds for $r \times r$ Hermitian matrices where $1 \leqslant r < n$. By the fundamental theorem of algebra, the characteristic polynomial c_M has a root λ in $\mathbb{C}$, which by the above is real. Let X denote the λ-eigenspace, that is

$$X = \{\mathbf{v} \in \mathbb{C}_n : M\mathbf{v} = \lambda\mathbf{v}\}.$$

Then X is a subspace as it is $\text{Null}(M - \lambda I)$ and has an orthonormal basis $\mathbf{v}_1, \ldots \mathbf{v}_m$ which we may extend to an orthonormal basis $\mathbf{v}_1, \ldots \mathbf{v}_n$ of $\mathbb{C}_n$. (All this is in the sense of the complex inner product $\bar{\mathbf{v}}^T\mathbf{w}$ of two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{C}_n$.

Let $U = (\mathbf{v}_1 | \ldots | \mathbf{v}_n)$; then U is unitary as $\overline{U}^T U = I_n$ is equivalent to the complex orthonormality of the columns. Then

$$\left[\overline{U}^T MU\right]_{ij} = \overline{\mathbf{v}_i}^T M\mathbf{v}_j$$

Note that $M\mathbf{v}_i = \lambda\mathbf{v}_i$ for $1 \leqslant i \leqslant m$. Also if $1 \leqslant i \leqslant m$ and $m < j, k \leqslant n$ we have that

$$\begin{aligned}
\overline{\mathbf{v}_i}^T M\mathbf{v}_j &= \overline{\mathbf{v}_i}^T\overline{M}^T\mathbf{v}_j = (\overline{M\mathbf{v}_i})^T\mathbf{v}_j = \lambda\overline{\mathbf{v}_i}^T\mathbf{v}_j = 0; \\
\overline{\mathbf{v}_k}^T(M\mathbf{v}_j) &= \overline{\mathbf{v}_k}^T\overline{M}^T\mathbf{v}_j = (\overline{M\mathbf{v}_k})^T\mathbf{v}_j = \overline{(\overline{\mathbf{v}_j}^T M\mathbf{v}_k)}^T
\end{aligned}$$

so that $\left[\overline{U}^T MU\right]_{ij} = 0$ and $\left[\overline{U}^T MU\right]_{kj} = \overline{\left[\overline{U}^T MU\right]_{jk}}$. This means that

$$\overline{U}^T MU = \text{diag}\,(\lambda I_m, N)$$

where N is a Hermitian $(n - m) \times (n - m)$ matrix. By our inductive hypothesis, there is a unitary $(n - m) \times (n - m)$ matrix V such that $\overline{V}^T NV$ is diagonal. If we set $W = \text{diag}\,(I_m, V)$ then W is unitary, UW is unitary and

$$(\overline{UW})^T M(UW) = \overline{W}^T\overline{U}^T MUW = \text{diag}\left(I_m, \bar{V}^T\right)\,\text{diag}\,(\lambda I_m, N)\,\text{diag}\,(I_m, V) = \text{diag}\left(\lambda I_m, \overline{V}^T NV\right)$$

is diagonal. This concludes the proof by induction.

Say that X is skew-symmetric and set $M = iX$. Then

$$\overline{M}^T = (-iX)^T = -iX^T = -i(-X) = iX = M$$

and so M is Hermitian. By the above proof $c_M(x)$ has real roots. The roots $c_X(x)$ are $\frac{1}{i}$ times the roots of $c_M(x)$ and so are purely imaginary.