Solution (#1226) Given a quaternion $Q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ we write $\overline{Q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$ and set $|Q|^2 = Q\overline{Q}$.
(i) Let $q = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. Note that

$$\begin{aligned} q^2 &= (b\mathbf{i} + c\mathbf{j} + d\mathbf{k})(b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) \\ &= b^2\mathbf{ii} + c^2\mathbf{jj} + d^2\mathbf{kk} + bc(\mathbf{ij} + \mathbf{ji}) + bd(\mathbf{ik} + \mathbf{ki}) + cd(\mathbf{jk} + \mathbf{kj}) \\ &= -b^2 - c^2 - d^2. \end{aligned}$$

Let

$$H = \frac{q}{\sqrt{b^2 + c^2 + d^2}},$$

so that $H^2 = -1$. Then

$$Q = a + \sqrt{b^2 + c^2 + d^2}\,H.$$

If we set $r = |Q| = \sqrt{a^2 + b^2 + c^2 + d^2}$, then we can write

$$Q = r(\cos\theta + H\sin\theta)$$

for some θ as

$$\left(\frac{a}{\sqrt{a^2 + b^2 + c^2 + d^2}}\right)^2 + \left(\frac{\sqrt{b^2 + c^2 + d^2}}{\sqrt{a^2 + b^2 + c^2 + d^2}}\right)^2 = 1.$$

(ii) If $P_1 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $P_2 = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ then

$$\begin{aligned} P_1P_2 &= (xX\mathbf{i}^2 + yY\mathbf{j}^2 + zZ\mathbf{k}^2) + (xY\mathbf{ij} + yZ\mathbf{jk} + zX\mathbf{ki}) + (yX\mathbf{ji} + zY\mathbf{kj} + xZ\mathbf{ik}) \\ &= -(xX + yY + zZ) + (yZ - zY)\mathbf{i} + (zX - xZ)\mathbf{j} + (xY - yX)\mathbf{k} \\ &= -P_1 \cdot P_2 + P_1 \wedge P_2. \end{aligned}$$

(iii) Let $Q = \sqrt{3}/2 + (\mathbf{i} + \mathbf{j} + \mathbf{k})/(2\sqrt{3})$ and $P = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$. Then

$$|Q|^2 = \frac{3}{4} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = 1.$$

Further by (ii) we have

$$\begin{aligned} QP\overline{Q} &= \left(\frac{\sqrt{3}}{2} + \frac{P}{2}\right) P \left(\frac{\sqrt{3}}{2} - \frac{P}{2}\right) = \left(\frac{\sqrt{3}}{2}P - \frac{1}{2}\right)\left(\frac{\sqrt{3}}{2} - \frac{P}{2}\right) \\ &= \left(\frac{3}{4} + \frac{1}{4}\right) P + \left(\frac{-\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\right) = P. \end{aligned}$$

If R is a unit vector perpendicular to P then

$$\begin{aligned} QR\overline{Q} &= \left(\frac{\sqrt{3}}{2} + \frac{P}{2}\right) R \left(\frac{\sqrt{3}}{2} - \frac{P}{2}\right) \\ &= \frac{1}{4}\left(\sqrt{3} + P\right) R \left(\sqrt{3} - P\right) \\ &= \frac{1}{4}\left(\sqrt{3}R + P \wedge R\right)\left(\sqrt{3} - P\right) \\ &= \frac{1}{4}\left(3R + \sqrt{3}P \wedge R - \sqrt{3}R \wedge P - (P \wedge R) \wedge P\right) \\ &= \frac{1}{2}R + \frac{\sqrt{3}}{2}P \wedge R \qquad [\text{as } (P \wedge R) \wedge P = R] \\ &= \left(\cos\frac{\pi}{3}\right) R + \left(\sin\frac{\pi}{3}\right) P \wedge R. \end{aligned}$$

Note $P, R, R \wedge P$ form an orthonormal basis. We see from the above calculations that T fixes the line through the origin parallel to P, which has equation $x = y = z$, and rotates the plane generated by R and $R \wedge P$ by $\pi/3$.