Solution (#1227) Any quaternion q can be uniquely written in the form

$$q = z_1 + z_2\mathbf{j}$$

where z_1 and z_2 are complex numbers of the form $x + y\mathbf{i}$. We can then identify q with the complex matrix

$$q \leftrightarrow Q = \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix}.$$

(i) Say $q_1 = z_1 + z_2\mathbf{j} \leftrightarrow Q_1$ and $q_2 = w_1 + w_2\mathbf{j} \leftrightarrow Q_2$. Then

$$q_1 + q_2 = (z_1 + w_1) + (z_2 + w_2)\mathbf{j} \leftrightarrow \begin{pmatrix} z_1 + w_1 & z_2 + w_2 \\ -z_2 + w_2 & z_1 + w_1 \end{pmatrix} = Q_1 + Q_2.$$

Also

$$q_1 q_2 = (z_1 + z_2\mathbf{j})(w_1 + w_2\mathbf{j}) = z_1 w_1 + z_2\mathbf{j}w_1 + z_1 w_2\mathbf{j} + z_2\mathbf{j}w_2\mathbf{j}.$$

Now for any complex number $z = a + b\mathbf{i}$ note that

$$\mathbf{j}z = \mathbf{j}(a + b\mathbf{i}) = a\mathbf{j} - b\mathbf{k} = (a - b\mathbf{i})\mathbf{j} = \bar{z}\mathbf{j}.$$

So

$$\begin{aligned} q_1 q_2 &= (z_1 w_1 - z_2\overline{w_2}) + (z_2\overline{w_1} + z_1 w_2)\mathbf{j} \\ &\leftrightarrow \begin{pmatrix} z_1 w_1 - z_2\overline{w_2} & z_2\overline{w_1} + z_1 w_2 \\ -\overline{z_2}w_1 - \overline{z_1}\overline{w_2} & \overline{z_1}\overline{w_1} - \overline{z_2}w_2 \end{pmatrix} \\ &= \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} \begin{pmatrix} w_1 & w_2 \\ -\overline{w_2} & \overline{w_1} \end{pmatrix} = Q_1 Q_2. \end{aligned}$$

The determinant of Q_1 equals $|z_1|^2 + |z_2|^2$ which is non-zero unless $z_1 = z_2 = 0$, i.e. unless $q_1 = 0$. If $q_1 \neq 0$ then

$$(Q_1)^{-1} = \frac{1}{|z_1|^2 + |z_2|^2} \begin{pmatrix} \overline{z_1} & -z_2 \\ \overline{z_2} & z_1 \end{pmatrix} \leftrightarrow \frac{\overline{z_1} - z_2\mathbf{j}}{|z_1|^2 + |z_2|^2} = \frac{1}{q_1}$$

as

$$(\overline{z_1} - z_2\mathbf{j})(z_1 + z_2\mathbf{j}) = \overline{z_1}z_1 - z_2\overline{z_1}\mathbf{j} + \overline{z_1}z_2\mathbf{j} + z_2\overline{z_2} = |z_1|^2 + |z_2|^2.$$

(ii) To show that the octonions do not associate we need only find three octonions O_1, O_2, O_3 such that

$$O_1(O_2O_3) \neq (O_1O_2)O_3.$$

We choose

$$O_1 = (\mathbf{i}, 0), \qquad O_2 = (0, \mathbf{i}), \qquad O_3 = (\mathbf{j}, 0)$$

Then recalling $(q_1, q_2)(r_1, r_2) = (q_1 r_1 - \overline{r_2}q_2, r_2 q_1 + q_2\overline{r_1})$ we note

$$\begin{aligned} O_1(O_2O_3) &= (\mathbf{i}, 0)\left[(0, \mathbf{i})(\mathbf{j}, 0)\right] = (\mathbf{i}, 0)(0, \mathbf{i}(-\mathbf{j})) = (\mathbf{i}, 0)(0, -\mathbf{k}) = (0, -\mathbf{ki}) = (0, -\mathbf{j}); \\ (O_1O_2)O_3 &= \left[(\mathbf{i}, 0)(0, \mathbf{i})\right](\mathbf{j}, 0) = (0, \mathbf{ii})(\mathbf{j}, 0) = (0, -1)(\mathbf{j}, 0) = (0, -1(-\mathbf{j})) = (0, \mathbf{j}). \end{aligned}$$