Solution (#1228) A vector $\mathbf{x}=(x_1,\ldots,x_7)$ can be identified with a "pure" octonion by setting $q_1=x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k}$ and $q_2=x_4+x_5\mathbf{i}+x_6\mathbf{j}+x_7\mathbf{k}$. Let $\mathbf{x},\mathbf{y}$ be two vectors in $\mathbb{R}^7$ which we identify with the octonions

$$X=(\mathbf{p}_1,x_4+\mathbf{p}_2),\qquad Y=(\mathbf{P}_1,y_4+\mathbf{P}_2),$$

where

$$\mathbf{p}_1=x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k},\qquad \mathbf{p}_2=x_5\mathbf{i}+x_6\mathbf{j}+x_7\mathbf{k},\qquad \mathbf{P}_1=y_1\mathbf{i}+y_2\mathbf{j}+y_3\mathbf{k},\qquad \mathbf{P}_2=y_5\mathbf{i}+y_6\mathbf{j}+y_7\mathbf{k},$$

are pure quaternions. Then

$$\begin{aligned} XY+(\mathbf{x}\cdot\mathbf{y}) &= (\mathbf{p}_1,x_4+\mathbf{p}_2)(\mathbf{P}_1,y_4+\mathbf{P}_2)+(\mathbf{x}\cdot\mathbf{y},0)\\ &= (\mathbf{p}_1\mathbf{P}_1-(y_4-\mathbf{P}_2)(x_4+\mathbf{p}_2)+\mathbf{x}\cdot\mathbf{y},(y_4+\mathbf{P}_2)\mathbf{p}_1+(x_4+\mathbf{p}_2)(-\mathbf{P}_1)). \end{aligned}$$

The above will represent a pure octonion if the first co-ordinate is a pure quaternion. Now that co-ordinate equals

$$\mathbf{p}_1\mathbf{P}_1-(y_4-\mathbf{P}_2)(x_4+\mathbf{p}_2)+\mathbf{x}\cdot\mathbf{y}=\mathbf{p}_1\mathbf{P}_1-x_4y_4+x_4\mathbf{P}_2-y_4\mathbf{p}_2+\mathbf{P}_2\mathbf{p}_2+\mathbf{x}\cdot\mathbf{y}$$

which by #1226(ii) has real part

$$-\mathbf{p}_1\cdot\mathbf{P}_1-x_4y_4-\mathbf{p}_2\cdot\mathbf{P}_2+\mathbf{x}\cdot\mathbf{y}=0.$$

Hence $\mathbf{x}\wedge\mathbf{y}=\mathbf{xy}+(\mathbf{x}\cdot\mathbf{y},0)$ defines a product yielding an output in $\mathbb{R}^7$ from two inputs in $\mathbb{R}^7$.

So if $\mathbf{x}=(\mathbf{p}_1,x,\mathbf{p}_2)$ and $\mathbf{y}=(\mathbf{P}_1,y,\mathbf{P}_2)$ then

$$\mathbf{x}\wedge\mathbf{y}=(\mathbf{p}_1\wedge\mathbf{P}_1+x\mathbf{P}_2-y\mathbf{p}_2+\mathbf{P}_2\wedge\mathbf{p}_2,\mathbf{p}_2\cdot\mathbf{P}_1-\mathbf{p}_1\cdot\mathbf{P}_2,y\mathbf{p}_1+\mathbf{P}_2\wedge\mathbf{p}_1-x\mathbf{P}_1-\mathbf{p}_2\wedge\mathbf{P}_1).$$

(a) Given the anti-symmetry of the vector product in $\mathbb{R}_3$ and the anti-symmetry of the above expressions we have $\mathbf{x}\wedge\mathbf{y}=-\mathbf{y}\wedge\mathbf{x}$.(b) We then have $(\mathbf{x}\wedge\mathbf{y})\cdot\mathbf{x}$ equals

$$\begin{aligned} &(\mathbf{p}_1\wedge\mathbf{P}_1+x\mathbf{P}_2-y\mathbf{p}_2+\mathbf{P}_2\wedge\mathbf{p}_2)\cdot\mathbf{p}_1+(\mathbf{p}_2\cdot\mathbf{P}_1-\mathbf{p}_1\cdot\mathbf{P}_2)x+(y\mathbf{p}_1+\mathbf{P}_2\wedge\mathbf{p}_1-x\mathbf{P}_1-\mathbf{p}_2\wedge\mathbf{P}_1)\cdot\mathbf{p}_2\\ =\ &(x\mathbf{P}_2-y\mathbf{p}_2+\mathbf{P}_2\wedge\mathbf{p}_2)\cdot\mathbf{p}_1+(\mathbf{p}_2\cdot\mathbf{P}_1-\mathbf{p}_1\cdot\mathbf{P}_2)x+(y\mathbf{p}_1+\mathbf{P}_2\wedge\mathbf{p}_1-x\mathbf{P}_1)\cdot\mathbf{p}_2\\ =\ &x(\mathbf{P}_2\cdot\mathbf{p}_1+\mathbf{p}_2\cdot\mathbf{P}_1-\mathbf{p}_1\cdot\mathbf{P}_2-\mathbf{P}_1\cdot\mathbf{p}_2)+y(-\mathbf{p}_2\cdot\mathbf{p}_1+\mathbf{p}_1\cdot\mathbf{p}_2)+[\mathbf{P}_2,\mathbf{p}_2,\mathbf{p}_1]+[\mathbf{P}_2,\mathbf{p}_1,\mathbf{p}_2]=0. \end{aligned}$$

We then also have $(\mathbf{x}\wedge\mathbf{y})\cdot\mathbf{y}=-(\mathbf{y}\wedge\mathbf{x})\cdot\mathbf{y}=-0=0.$

(c) The linearity of the product in $\mathbf{x}$ is immediate.

(d) Finally let $\mathbf{x},\mathbf{y}$ be perpendicular unit vectors. This means that

$$1=|\mathbf{p}_1|^2+|\mathbf{p}_2|^2+x^2=|\mathbf{P}_1|^2+|\mathbf{P}_2|^2+y^2;\qquad 0=\mathbf{p}_1\cdot\mathbf{P}_1+xy+\mathbf{p}_2\cdot\mathbf{P}_2.$$

We have

$$|\mathbf{x}\wedge\mathbf{y}|^2=|\mathbf{p}_1\wedge\mathbf{P}_1+x\mathbf{P}_2-y\mathbf{p}_2+\mathbf{P}_2\wedge\mathbf{p}_2|^2+(\mathbf{p}_2\cdot\mathbf{P}_1-\mathbf{p}_1\cdot\mathbf{P}_2)^2+|y\mathbf{p}_1+\mathbf{P}_2\wedge\mathbf{p}_1-x\mathbf{P}_1-\mathbf{p}_2\wedge\mathbf{P}_1|^2.$$

Now $|\mathbf{p}_1\wedge\mathbf{P}_1+x\mathbf{P}_2-y\mathbf{p}_2+\mathbf{P}_2\wedge\mathbf{p}_2|^2$ equals

$$|\mathbf{p}_1\wedge\mathbf{P}_1|^2+|\mathbf{P}_2\wedge\mathbf{p}_2|^2+x^2|\mathbf{P}_2|^2+y^2|\mathbf{p}_2|^2+2(\mathbf{p}_1\wedge\mathbf{P}_1)\cdot(\mathbf{P}_2\wedge\mathbf{p}_2)+2x[\mathbf{p}_1,\mathbf{P}_1,\mathbf{P}_2]-2y[\mathbf{p}_1,\mathbf{P}_1,\mathbf{p}_2]-2xy\mathbf{P}_2\cdot\mathbf{p}_2.$$

Similarly $|y\mathbf{p}_1+\mathbf{P}_2\wedge\mathbf{p}_1-x\mathbf{P}_1-\mathbf{p}_2\wedge\mathbf{P}_1|^2$ equals

$$|\mathbf{P}_2\wedge\mathbf{p}_1|^2+|\mathbf{p}_2\wedge\mathbf{P}_1|^2+x^2|\mathbf{P}_1|^2+y^2|\mathbf{p}_1|^2-2(\mathbf{P}_2\wedge\mathbf{p}_1)\cdot(\mathbf{p}_2\wedge\mathbf{P}_1)-2x[\mathbf{P}_2,\mathbf{p}_1,\mathbf{P}_1]-2y[\mathbf{p}_2,\mathbf{P}_1,\mathbf{p}_1]-2xy\mathbf{p}_1\cdot\mathbf{P}_1.$$

By the quadruple scalar product we have

$$\begin{aligned} (\mathbf{p}_1\wedge\mathbf{P}_1)\cdot(\mathbf{P}_2\wedge\mathbf{p}_2) &= (\mathbf{p}_1\cdot\mathbf{P}_2)(\mathbf{P}_1\cdot\mathbf{p}_2)-(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{P}_1\cdot\mathbf{P}_2)\\ (\mathbf{P}_2\wedge\mathbf{p}_1)\cdot(\mathbf{p}_2\wedge\mathbf{P}_1) &= (\mathbf{p}_1\cdot\mathbf{P}_1)(\mathbf{P}_2\cdot\mathbf{p}_2)-(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{P}_1\cdot\mathbf{P}_2) \end{aligned}$$

and

$$\begin{aligned} |\mathbf{p}_1\wedge\mathbf{P}_1|^2 &= |\mathbf{p}_1|^2|\mathbf{P}_1|^2-(\mathbf{p}_1\cdot\mathbf{P}_1)^2, \qquad |\mathbf{P}_2\wedge\mathbf{p}_2|^2=|\mathbf{p}_2|^2|\mathbf{P}_2|^2-(\mathbf{p}_2\cdot\mathbf{P}_2)^2\\ |\mathbf{P}_2\wedge\mathbf{p}_1|^2 &= |\mathbf{p}_1|^2|\mathbf{P}_2|^2-(\mathbf{p}_1\cdot\mathbf{P}_2)^2, \qquad |\mathbf{p}_2\wedge\mathbf{P}_1|^2=|\mathbf{p}_2|^2|\mathbf{P}_1|^2-(\mathbf{p}_2\cdot\mathbf{P}_1)^2. \end{aligned}$$

So we see $|\mathbf{x}\wedge\mathbf{y}|^2$ equals with simplification

$$\begin{aligned} &(|\mathbf{p}_1|^2+|\mathbf{p}_2|^2)(|\mathbf{P}_2|^2+|\mathbf{P}_1|^2)-(\mathbf{p}_1\cdot\mathbf{P}_1+\mathbf{p}_2\cdot\mathbf{P}_2)^2+x^2(1-y^2)+y^2(1-x^2)-2xy(\mathbf{p}_1\cdot\mathbf{P}_1+\mathbf{p}_2\cdot\mathbf{P}_2)\\ =\ &(1-x^2)(1-y^2)-(-xy)^2+x^2(1-y^2)+y^2(1-x^2)-2xy(-xy)\\ =\ &1-x^2-y^2+x^2y^2-x^2y^2+x^2-x^2y^2+y^2-x^2y^2+2x^2y^2=1 \end{aligned}$$

as required, thus showing that $\mathbf{x}\wedge\mathbf{y}$ is a unit vector.