Solution (#1232) Let X be a subspace of $\mathbb{R}^n$. Say that $\mathbf{v}=\mathbf{x}+\mathbf{y}$ is the decomposition of $\mathbf{v}$ into $\mathbf{x}$ in X and $\mathbf{y}$ in $X^{\perp}$.

(i) Let $\mathbf{x}'$ be any other point of X. Then

$$\left|\mathbf{v}-\mathbf{x}'\right|^2=\left|\mathbf{x}-\mathbf{x}'+\mathbf{y}\right|^2=\left|\mathbf{x}-\mathbf{x}'\right|^2+\left|\mathbf{y}\right|^2\geqslant\left|\mathbf{y}\right|^2$$

as $\mathbf{x}-\mathbf{x}'$ is perpendicular to $\mathbf{y}$. We can see then that $|\mathbf{v}-\mathbf{x}'|$ is minimized when $\mathbf{x}'=\mathbf{x}$.

(ii) Say now that

$$\mathbf{v}_1=\mathbf{x}_1+\mathbf{y}_1 \qquad \text{and} \qquad \mathbf{v}_2=\mathbf{x}_2+\mathbf{y}_2$$

are two such decompositions and also α_1,α_2 are real numbers. Then

$$\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2=(\alpha_1\mathbf{x}_1+\alpha_2\mathbf{x}_2)+(\alpha_1\mathbf{y}_1+\alpha_2\mathbf{y}_2)\,.$$

As X and $X^{\perp}$ are both subspaces then $\alpha_1\mathbf{x}_1+\alpha_2\mathbf{x}_2$ is in X and $\alpha_1\mathbf{y}_1+\alpha_2\mathbf{y}_2$ is in $X^{\perp}$. So

$$P(\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2)=\alpha_1\mathbf{x}_1+\alpha_2\mathbf{x}_2$$

and we see that P is linear.

Finally the image of P is in X by definition, and if $\mathbf{x}$ is in X then

$$\mathbf{x}=\mathbf{x}+\mathbf{0}$$

is its unique decomposition and so $P\mathbf{x}=\mathbf{x}$ by definition. If $\mathbf{y}$ is in $X^{\perp}$ then

$$\mathbf{y}=\mathbf{0}+\mathbf{y}$$

is the unique decomposition and so $P\mathbf{y}=\mathbf{0}$. Conversely if $P\mathbf{v}=0$ then it follows that

$$\mathbf{v}=\mathbf{0}+\mathbf{v}$$

is the decomposition of $\mathbf{v}$ and hence $\mathbf{v}$ is in $X^{\perp}$.