Solution (#1233) Let A be an $m \times n$ matrix. Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be the standard basis of $\mathbb{R}^m$.

(i) $(\mathrm{Row}(A))^{\perp} = \mathrm{Null}(A)$.

Let $\mathbf{v}$ in $\mathbb{R}_n$ be in $\mathrm{Null}(A)$ so that $A\mathbf{v} = \mathbf{0}$. The ith row of A is $\mathbf{e}_i A$ and so we have that

$$\mathbf{e}_i A \cdot \mathbf{v} = (\mathbf{e}_i A)\mathbf{v} = \mathbf{e}_i (A\mathbf{v}) = \mathbf{0}.$$

As the rows of A span $\mathrm{Row}(A)$ then $\mathrm{Null}(A)$ is contained in $(\mathrm{Row}(A))^{\perp}$.

Conversely if $\mathbf{v}$ is in $(\mathrm{Row}(A))^{\perp}$ then $\mathbf{v}$ is perpendicular to the ith row $\mathbf{e}_i A$ of A. It again follows that

$$\mathbf{e}_i (A\mathbf{v}) = \mathbf{e}_i A \cdot \mathbf{v} = \mathbf{0}$$

for each i and hence that $A\mathbf{v} = \mathbf{0}$. Hence $\mathbf{v}$ is in $\mathrm{Null}(A)$.

(ii) Applying (i) to A^T we then have

$$(\mathrm{Col}(A))^{\perp} = (\mathrm{Row}(A^T))^{\perp} = \mathrm{Null}(A^T)$$

as required.