Solution (#1236) Recall by the Cayley-Hamilton theorem that $m_M(x)$ divides $c_M(x)$ for any square matrix M

Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then $c_A(x) = x^3$ and so $m_A(x)$ could be x, x^2 or x^3. As

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$$

then $m_A(x) = x^3 = c_A(x)$.

Note that

$$c_B(x) = \begin{vmatrix} x & -1 & 0 \\ 0 & x & -1 \\ -1 & 0 & x \end{vmatrix} = x^3 - 1.$$

We can also see that

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad B^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

are linearly independent (by focusing on the first row) and so $m_B(x) = x^3 - 1 = c_B(x)$.

Finally

$$c_C(x) = \begin{vmatrix} x-2 & 0 & 0 \\ 0 & x-2 & -1 \\ 0 & 0 & x-2 \end{vmatrix} = (x-2)^3.$$

So $m_C(x)$ could be $x - 2$, $(x-2)^2$ or $(x-2)^3$. Now $C - 2I \neq 0$ but

$$(C - 2I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 = 0_{33}$$

and hence $m_C(x) = (x-2)^2 \neq c_C(x)$. Consequently there is no C-cyclic vector by #1235.

To find a cyclic vector for A we note

$$\mathbf{e}_3^T, \qquad A\mathbf{e}_3^T = \mathbf{e}_1^T + \mathbf{e}_2^T, \qquad A^2\mathbf{e}_3^T = \mathbf{e}_1^T$$

is a basis.

To find a cyclic vector for B we note

$$\mathbf{e}_1^T, \qquad B\mathbf{e}_1^T = \mathbf{e}_3^T, \qquad B^2\mathbf{e}_1^T = \mathbf{e}_2^T$$

is a basis.