Solution (#1246) (i) As $|u| < c$ we can write $u = c \tanh \phi$ for a unique ϕ. Then

$$\gamma(u) = \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{1}{\sqrt{1 - \tanh^2 \phi}} = \cosh \phi.$$

Hence

$$L(u) = \gamma(u) \begin{pmatrix} 1 & u/c \\ u/c & 1 \end{pmatrix} = \cosh \phi \begin{pmatrix} 1 & \tanh \phi \\ \tanh \phi & 1 \end{pmatrix} = \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix},$$

and note any matrix of this form is a Lorentz matrix. Now for real ϕ and ψ we have

$$\begin{aligned} & \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix} \begin{pmatrix} \cosh \psi & \sinh \psi \\ \sinh \psi & \cosh \psi \end{pmatrix} \\ = & \begin{pmatrix} \cosh \phi \cosh \psi + \sinh \phi \sinh \psi & \cosh \phi \sinh \psi + \sinh \phi \cosh \psi \\ \sinh \phi \cosh \psi + \cosh \phi \sinh \psi & \sinh \phi \sinh \psi + \cosh \phi \cosh \psi \end{pmatrix} \\ = & \begin{pmatrix} \cosh(\phi + \psi) & \sinh(\phi + \psi) \\ \sinh(\phi + \psi) & \cosh(\phi + \psi) \end{pmatrix} \end{aligned}$$

which is another Lorentz matrix, and we can also note

$$\begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix}^{-1} = \begin{pmatrix} \cosh(-\phi) & \sinh(-\phi) \\ \sinh(-\phi) & \cosh(-\phi) \end{pmatrix} = \begin{pmatrix} \cosh \phi & -\sinh \phi \\ -\sinh \phi & \cosh \phi \end{pmatrix},$$

is a Lorentz matrix.
(ii) Let $g = \operatorname{diag}(1, -1)$ and let

$$L = \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix}.$$

Then

$$\begin{aligned} L^T g L &= \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix} \\ &= \begin{pmatrix} \cosh \phi & -\sinh \phi \\ \sinh \phi & -\cosh \phi \end{pmatrix} \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix} \\ &= \begin{pmatrix} \cosh^2 \phi - \sinh^2 \phi & \cosh \phi \sinh \phi - \sinh \phi \cosh \phi \\ \sinh \phi \cosh \phi - \cosh \phi \sinh \phi & \sinh^2 \phi - \cosh^2 \phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = g. \end{aligned}$$

(iii) Say now that M is a 2×2 matrix such that

$$M^T g M = g, \qquad [M]_{11} > 0, \qquad \det M > 0.$$

If we write

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $M^T g M$ equals

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -c \\ b & -d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 - c^2 & ab - dc \\ ab - dc & b^2 - d^2 \end{pmatrix}.$$

So if $M^T g M = g$ then

$$a^2 - c^2 = 1 = d^2 - b^2.$$

So we can write $a = \pm \cosh \phi$, $c = \sinh \phi$ and $d = \pm \cosh \psi$, $b = \sinh \psi$ for some unique ϕ, ψ. As $a = [M]_{11} > 0$ then $a = \cosh \phi$. Further as

$$\begin{aligned} 0 < \det M = & \quad ad - bc \\ = & \quad \pm \cosh \phi \cosh \psi - \sinh \psi \sinh \phi \\ = & \quad \pm \cosh (\phi \mp \psi), \end{aligned}$$

then we need to take the positive choice and so $d = \cosh \psi$. Finally as

$$0 = ab - dc = \cosh \phi \sinh \psi - \cosh \psi \sinh \phi = \sinh (\psi - \phi)$$

and so $\phi = \psi$. So

$$M = \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix}$$

and M is indeed a Lorentz matrix.