Solution (#1247) (i) The co-ordinates of two observers O and O' are related by

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma(u) \begin{pmatrix} 1 & u/c \\ u/c & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} \tag{11.10}$$

where $|u| < c$. Say that $u/c \approx 0$ so that $\gamma(u) \approx 1$. Then

$$\begin{aligned} ct &= ct' + (u/c)x', \qquad x = (u/c)(ct') + x', \\ \begin{pmatrix} ct \\ x \end{pmatrix} &= \begin{pmatrix} 1 & u/c \\ u/c & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} \end{aligned}$$

which can be rewritten

$$t \approx t', \qquad x = ut' + x',$$

which is approximately the Galilean transformation

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}.$$

(ii) Recall that $x' = 0$ at O'at all times. So the (ct, x)-co-ordinates of O' are

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma(u) \begin{pmatrix} 1 & u/c \\ u/c & 1 \end{pmatrix} \begin{pmatrix} ct' \\ 0 \end{pmatrix} = \gamma(u) \begin{pmatrix} ct' \\ ut' \end{pmatrix}.$$

Hence we see

$$x = \gamma u t' = u(\gamma t') = ut$$

and the velocity of O' as measure by O is u.

(iii) We have

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma(u) \begin{pmatrix} 1 & u/c \\ u/c & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & v/c \\ v/c & 1 \end{pmatrix} \begin{pmatrix} ct'' \\ x'' \end{pmatrix}.$$

Then

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma(u)\gamma(v) \begin{pmatrix} 1 & u/c \\ u/c & 1 \end{pmatrix} \begin{pmatrix} 1 & v/c \\ v/c & 1 \end{pmatrix} \begin{pmatrix} ct'' \\ x'' \end{pmatrix}.$$

Note that

$$\begin{pmatrix} 1 & u/c \\ u/c & 1 \end{pmatrix} \begin{pmatrix} 1 & v/c \\ v/c & 1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{uv}{c^2} & \frac{u+v}{c} \\ \frac{u+v}{c} & 1 + \frac{uv}{c^2} \end{pmatrix}$$

and that

$$\begin{aligned} \gamma \left(\frac{u+v}{1+uv/c^2} \right)^{-2} &= 1 - \frac{(u+v)^2}{(1+uv/c^2)^2 c^2} \\ &= \frac{c^4 + u^2v^2 - u^2c^2 - v^2c^2}{(c^2+uv)^2} \\ &= \frac{c^4(1-u^2/c^2)(1-v^2/c^2)}{(c^2+uv)^2} \\ &= \frac{\gamma(u)^{-2}\gamma(v)^{-2}}{(1+uv/c^2)^2}. \end{aligned}$$

Hence

$$\begin{aligned} \gamma(u)\gamma(v) \begin{pmatrix} 1 & u/c \\ u/c & 1 \end{pmatrix} \begin{pmatrix} 1 & v/c \\ v/c & 1 \end{pmatrix} &= \gamma \left(\frac{u+v}{1+uv/c^2} \right) \left(1 + \frac{uv}{c^2} \right)^{-1} \begin{pmatrix} 1 + \frac{uv}{c^2} & \frac{u+v}{c} \\ \frac{u+v}{c} & 1 + \frac{uv}{c^2} \end{pmatrix} \\ &= \gamma \left(\frac{u+v}{1+uv/c^2} \right) \begin{pmatrix} 1 & \frac{1}{c}\left(\frac{u+v}{1+uv/c^2} \right) \\ \frac{1}{c}\left(\frac{u+v}{1+uv/c^2} \right) & 1 \end{pmatrix}. \end{aligned}$$

So O'' is moving with velocity

$$\frac{u+v}{1+uv/c^2}$$

relative to O. Note that when $v = c$ then

$$\frac{u+v}{1+uv/c^2} = \frac{u+c}{1+u/c} = c.$$

This agrees with the idea that all observers agree on the speed of light, irrespective of their relative velocities.