Solution (#1248) (i) The co-ordinates of two observers O and O' are related by

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma(u) \begin{pmatrix} 1 & u/c \\ u/c & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

where $|u| < c$.

The events $(ct', x') = (0, 0)$ and $(ct', x') = (0, 1)$ are perceived as simultaneous by O'. However O assigns times

$$ct = 0 \qquad \text{and} \qquad ct = \gamma(u)u/c$$

which are not equal and so the events not perceived as simultaneous.

(ii) Say that O' is at $E_1 = (ct'_1, 0)$ and $E_2 = (ct'_2, 0)$ at two different events. Then O assigns these events t-co-ordinates of

$$ct_1 = \gamma(u)ct'_1, \qquad ct_2 = \gamma(u)ct'_2.$$

So the time difference as measure by O is

$$t_2 - t_1 = \gamma(u)(t'_2 - t'_1).$$

Consequently, as $\gamma(u) > 1$ then O measures a greater time having passed than O' does. This phenomenon is known as time dilation.

(iii) Two twins O and O' meet at a common origin $(ct, x) = (ct', x') = (0, 0)$ on Earth. Twin O', an astronaut, then travels away from O at speed v to a planet distance d away. The time taken to do this, measured by O is d/v but, because of time dilation, the time measured by O' is

$$\frac{d}{v\gamma(v)}.$$

The astronaut perceives the same time $d/(v\gamma(v))$ on the return journey which O again measures as d/v. So the difference in ages when the twin returns is

$$\frac{2d}{v} - \frac{2d}{v\gamma(v)} = \frac{2d}{v}\left(1 - \sqrt{1 - v^2/c^2}\right)$$

which will be negligible unless v is comparable to the speed of light. By the binomial theorem (for an arbitrary exponent)

$$\sqrt{1 - \frac{v^2}{c^2}} = 1 + \frac{1}{2}\left(-\frac{v^2}{c^2}\right) + \frac{1}{2} \times -\frac{1}{2}\left(-\frac{v^2}{c^2}\right)^2 + \cdots \approx 1 - \frac{v^2}{2c^2}.$$

The above therefore is approximately

$$\frac{2d}{v}\left(1 - \sqrt{1 - v^2/c^2}\right) = \frac{2d}{v}\left(1 - 1 + \frac{v^2}{2c^2}\right) = \frac{dv}{c^2}$$

(iv) Observer O' has a rod of length L with its ends having co-ordinates $(ct', 0)$ and (ct', L). So O perceives these events as two non-simultaneous events

$$\begin{aligned} \gamma(u)\begin{pmatrix} 1 & u/c \\ u/c & 1 \end{pmatrix}\begin{pmatrix} ct' \\ 0 \end{pmatrix} &= \gamma(u)\begin{pmatrix} ct' \\ ut' \end{pmatrix}; \\ \gamma(u)\begin{pmatrix} 1 & u/c \\ u/c & 1 \end{pmatrix}\begin{pmatrix} ct' \\ L \end{pmatrix} &= \gamma(u)\begin{pmatrix} ct' + uL/c \\ ut' + L \end{pmatrix}. \end{aligned}$$

Given that O measures distances between simultaneous events, then O measures the distance between two events that aren't simultaneous in the rod's frame say when $t' = t'_1$ and $t' = t'_2$. So we first need

$$ct'_1 = ct'_2 + uL/c \qquad \text{or} \qquad t'_2 - t'_1 = -\frac{uL}{c^2}$$

The length of the rod is then measured as

$$\gamma(u)\left(ut'_2 + L - ut'_1\right) = \gamma(u)\left(L + u\left(-\frac{uL}{c^2}\right)\right) = \gamma(u)\left(1 - \frac{u^2}{c^2}\right)L = \frac{\gamma(u)L}{\gamma(u)^2} = \frac{L}{\gamma(u)}.$$