Solution (#1256) (i) Let ϕ be a step function on $[a,b]$. Then there are subintervals $J_1,\ldots,J_n$ and real numbers $c_1,\ldots,c_n$ such that

$$\phi = \sum_{k=1}^{n} c_k \mathbf{1}_{J_k}.$$

Let $a_1,a_2,\ldots,a_m$ be the points where ϕ changes value – there can only be finitely many such points as ϕ can only change value at an end of a subinterval J_i. The function ϕ is constant on each of the subintervals

$$[a_i,a_i] \qquad \text{and} \qquad (a_i,a_{i+1})$$

say taking values r_i and s_i and these subintervals are disjoint by construction. So

$$\phi = \sum_{k=1}^{m} r_k \mathbf{1}_{[a_i,a_i]} + \sum_{k=1}^{m-1} s_k \mathbf{1}_{(a_i,a_{i+1})}$$

as desired.

(ii) Now let ϕ and ψ be step functions on $[a,b]$. There are finitely many points $b_1,b_2,\ldots,b_m$ be the points where ϕ or ψ change value. Thus ϕ and ψ are both constant on each of the subintervals

$$[b_i,b_i] \qquad \text{and} \qquad (b_i,b_{i+1}).$$

So we may take these subintervals as $I_1,\ldots,I_N$ and we have

$$\phi = \sum_{k=1}^{N} y_k \mathbf{1}_{I_k} \quad \text{and} \quad \psi = \sum_{k=1}^{N} Y_k \mathbf{1}_{I_k}$$

for some scalars $y_1,\ldots,y_N,Y_1,\ldots,Y_N$.