Solution (#1261) (i) The given function equals

$$f(x) = (x+1)\mathbf{1}_{[-1,0)}(x) + (1-x)\mathbf{1}_{[0,1[}(x).$$

We define

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t.$$

$$\begin{array}{rl} \text{For } x \leqslant -1: & F(x) = 0. \\ \text{For } -1 \leqslant x \leqslant 0: & F(x) = \displaystyle\int_{-1}^{x} (1+t)\,\mathrm{d}t = \frac{1}{2}(1+x)^2. \\ \text{For } 0 \leqslant x \leqslant 1: & F(x) = F(0) + \displaystyle\int_{0}^{x} (1-t)\,\mathrm{d}t = \frac{1}{2} - \left[\frac{(1-t)^2}{2}\right]_0^x = \frac{1+2x-x^2}{2}. \\ \text{For } 1 \leqslant x: & F(x) = F(1) = 1. \end{array}$$

(ii) (a) By FTC we have

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = F(1) - F(-1) = 1 - 0 = 1.$$

(b) For the second integral

$$\begin{aligned} \int_{-1}^{1} f(x)^2\,\mathrm{d}x &= \int_{-1}^{0} (1+x)^2\,\mathrm{d}x + \int_{0}^{1} (1-x)^2\,\mathrm{d}x \\ &= \left[\frac{(1+x)^3}{3}\right]_{-1}^{0} - \left[\frac{(1-x)^3}{3}\right]_{0}^{1} \\ &= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}. \end{aligned}$$

(c) For x in the interval $[-1,1]$ we have that $-1 \leqslant x^2 - 1 \leqslant 0$ and so

$$f(x^2-1) = (x^2-1) + 1 = x^2.$$

Hence

$$\int_{-1}^{1} f(x^2-1)\,\mathrm{d}x = \int_{-1}^{1} x^2\,\mathrm{d}x = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}.$$

(d) For any real x we have $x^2 + 1 \geqslant 1$ and hence

$$f(x^2+1) = 0.$$

Hence

$$\int_{-\infty}^{\infty} f(x^2+1)\,\mathrm{d}x = 0.$$

(e) f is non-zero in the interval $[-1,1]$ and x^2 is in $[-1,1]$ for x in $[-1,1]$. In fact for x in $[-1,1]$ we see that x^2 is in $[0,1]$. Hence we have

$$\int_{-\infty}^{\infty} f(x^2)\,\mathrm{d}x = \int_{-1}^{1} 1 - x^2\,\mathrm{d}x = 2 - \frac{2}{3} = \frac{4}{3}.$$