Solution (#1271) Let f be an integrable function on $[a,b]$ and let $\varepsilon > 0$. Then there exist step functions ϕ and ψ such that

$$\phi(x) \leqslant f(x) \leqslant \psi(x) \qquad \text{and} \qquad 0 \leqslant \int_a^b (\psi(x) - \phi(x))\, \mathrm{d}x < \varepsilon.$$

Then $\min\{|\phi|, |\psi|\}$ and $\max\{|\phi|, |\psi|\}$ are step functions with

$$\min\{|\phi|, |\psi|\} \leqslant |f| \leqslant \max\{|\phi|, |\psi|\}$$

and by the triangle inequality

$$\max\{|\phi|, |\psi|\} - \min\{|\phi|, |\psi|\} = ||\psi| - |\phi|| \leqslant |\psi - \phi| = \psi - \phi,$$

so that

$$0 \leqslant \int_a^b (\max\{|\phi(x)|, |\psi(x)|\} - \min\{|\phi(x)|, |\psi(x)|\})\, \mathrm{d}x < \varepsilon.$$

Hence $|f|$ is integrable and further as

$$-|f| \leqslant f \leqslant |f|$$

then by Proposition 5.12(b) we have

$$-\int_a^b |f|\, \mathrm{d}x \leqslant \int_a^b f\, \mathrm{d}x \leqslant \int_a^b |f|\, \mathrm{d}x$$

so that

$$\left| \int_a^b f\, \mathrm{d}x \right| \leqslant \int_a^b |f|\, \mathrm{d}x.$$

Viewing integrals as "continuous" sums, this inequality might be seen as a continuous version of the discrete triangle inequality

$$\left| \sum_{i=1}^n a_i \right| \leqslant \sum_{i=1}^n |a_i|$$

for finite sums.

Finally if g is a second integrable function then $f - g$ is integrable and so therefore is $|f - g|$ by the above. We then need only note

$$\max\{f, g\} = \frac{f+g}{2} + \frac{|f-g|}{2}, \qquad \min\{f, g\} = \frac{f+g}{2} - \frac{|f-g|}{2}$$

are integrable.