Solution (#1272) Let f, g be integrable functions on $[a,b]$. For now assume that $f(x) \geqslant 0$ and $g(x) \geqslant 0$ for all $a \leqslant x \leqslant b$. As integrable functions are, by definition, bounded there exist $R, S > 0$ such that $0 \leqslant f \leqslant R$ and $0 \leqslant g \leqslant S$.

Let $\varepsilon > 0$. There then exist step functions $\phi_1, \phi_2, \psi_1, \psi_2$ such that

$$\phi_1(x) \leqslant f(x) \leqslant \psi_1(x), \qquad \phi_2(x) \leqslant g(x) \leqslant \psi_2(x), \qquad \text{for all } a \leqslant x \leqslant b$$

and

$$0 \leqslant \int_a^b (\psi_1(x) - \phi_1(x))\, \mathrm{d}x < \frac{\varepsilon}{2(S+1)}, \qquad 0 \leqslant \int_a^b (\psi_2(x) - \phi_2(x))\, \mathrm{d}x < \frac{\varepsilon}{2R}.$$

(The reasoning behind these choices will become clearer towards the end of the proof.)

We may assume that the step functions ϕ_1 and ϕ_2 are themselves non-negative (if not we might replace ϕ_i with $\max\{0, \phi_i\}$); we may also assume that $\psi_2 \leqslant S+1$ (if not we might replace ψ_2 with $\min\{\psi_2, S+1\}$). So we have

$$\phi_1(x)\phi_2(x) \leqslant f(x)g(x) \leqslant \psi_1(x)\psi_2(x)$$

and also that

$$\int_a^b (\psi_1(x)\psi_2(x) - \phi_1(x)\phi_2(x))\, \mathrm{d}x = \int_a^b (\psi_1(x) - \phi_1(x))\, \psi_2(x)\, \mathrm{d}x + \int_a^b \phi_1(x)\, (\psi_2(x) - \phi_2(x))\, \mathrm{d}x.$$

Now by Proposition 5.6(b)

$$\begin{aligned} \left| \int_a^b (\psi_1(x) - \phi_1(x))\, \psi_2(x)\, \mathrm{d}x \right| &\leqslant (S+1) \left| \int_a^b (\psi_1(x) - \phi_1(x))\, \mathrm{d}x \right| < (S+1) \times \frac{\varepsilon}{2(S+1)} = \frac{\varepsilon}{2}; \\ \left| \int_a^b \phi_1(x)\, (\psi_2(x) - \phi_2(x))\, \mathrm{d}x \right| &\leqslant R \left| \int_a^b (\psi_2(x) - \phi_2(x))\, \mathrm{d}x \right| < R \times \frac{\varepsilon}{2R} = \frac{\varepsilon}{2}. \end{aligned}$$

Hence

$$0 \leqslant \int_a^b (\psi_1(x)\psi_2(x) - \phi_1(x)\phi_2(x))\, \mathrm{d}x < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We have shown above that the product of two non-negative integrable functions is integrable. Suppose more generally that f and g are integrable (and not necessarily non-negative). As f, g are bounded then there exist M and N such that $f + M \geqslant 0$ and $g + N \geqslant 0$. Then $f + M$ and $g + N$ are integrable and so is

$$(f+M)(g+N) = fg + Mg + Nf + MN$$

by our previous argument. But then by Proposition 5.12(a)

$$fg = (f+M)(g+N) - Mg - Nf - MN$$

is also integrable.