Solution (#1273) Let f be a non-negative integrable function on $[a,b]$. Let $\varepsilon > 0$. Then there are step functions ϕ and ψ such that

$$\phi(x) \leqslant f(x) \leqslant \psi(x) \qquad \text{for } a \leqslant x \leqslant b$$

with

$$0 \leqslant \int_a^b \psi(x)\,\mathrm{d}x - \int_a^b \phi(x)\,\mathrm{d}x < \frac{\varepsilon^2}{4(b-a)}.$$

Without loss of generality we can assume that ϕ is non-negative as well. (If not we can replace it with $\max\{0,\phi\}$.) Note that

$$\sqrt{\phi(x)} \leqslant \sqrt{f(x)} \leqslant \sqrt{\psi(x)}$$

and that $\sqrt{\phi(x)}$ and $\sqrt{\psi(x)}$ are step functions. We will consider two cases

$$\text{the subset } I \text{ where } \sqrt{\psi(x)} \leqslant \frac{\varepsilon}{2(b-a)}; \qquad \text{the subset } J \text{ where } \sqrt{\psi(x)} > \frac{\varepsilon}{2(b-a)}.$$

Then

$$\int_I \left(\sqrt{\psi(x)} - \sqrt{\phi(x)}\right)\,\mathrm{d}x \leqslant (b-a) \times \frac{\varepsilon}{2(b-a)} = \frac{\varepsilon}{2},$$

and

$$\begin{aligned}
\int_J \left(\sqrt{\psi(x)} - \sqrt{\phi(x)}\right)\,\mathrm{d}x &= \int_J \frac{\psi(x)-\phi(x)}{\sqrt{\psi(x)}+\sqrt{\phi(x)}}\,\mathrm{d}x \\
&\leqslant \int_J \frac{\psi(x)-\phi(x)}{\sqrt{\psi(x)}}\,\mathrm{d}x \\
&\leqslant \frac{2(b-a)}{\varepsilon}\int_J \psi(x)-\phi(x)\,\mathrm{d}x \\
&< \frac{2(b-a)}{\varepsilon} \times \frac{\varepsilon^2}{4(b-a)} = \frac{\varepsilon}{2}.
\end{aligned}$$

Hence

$$\int_a^b \left(\sqrt{\psi(x)} - \sqrt{\phi(x)}\right)\,\mathrm{d}x \leqslant \int_I \left(\sqrt{\psi(x)} - \sqrt{\phi(x)}\right)\,\mathrm{d}x + \int_J \left(\sqrt{\psi(x)} - \sqrt{\phi(x)}\right)\,\mathrm{d}x < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

showing that $\sqrt{f}$ is integrable.