**Solution** (#1280) Let $f, g$ be increasing, positive integrable functions on $[a,b]$. It follows then that

$$(f(x)-f(y))(g(x)-g(y)) \geqslant 0 \qquad \text{for all } x,y \text{ in } [a,b],$$

as the product involves two non-negative numbers when $x \geqslant y$ and two non-positive ones when $x < y$. So

$$\int_{x=a}^{x=b}\int_{y=a}^{y=b} (f(x)-f(y))(g(x)-g(y))\,\mathrm{d}y\,\mathrm{d}x \geqslant 0.$$

Expanding we find

$$\begin{aligned}&\left(\int_{x=a}^{x=b} f(x)g(x)\,\mathrm{d}x\right)\left(\int_{y=a}^{y=b}\mathrm{d}y\right) - \left(\int_{x=a}^{x=b} f(x)\,\mathrm{d}x\right)\left(\int_{y=a}^{y=b} g(y)\,\mathrm{d}y\right)\\ &-\left(\int_{x=a}^{x=b} g(x)\,\mathrm{d}x\right)\left(\int_{y=a}^{y=b} f(y)\,\mathrm{d}y\right)+\left(\int_{x=a}^{x=b}\mathrm{d}x\right)\left(\int_{y=a}^{y=b} f(y)g(y)\,\mathrm{d}y\right) \quad\geqslant\quad 0\end{aligned}$$

which rearranges to the desired inequality

$$\left(\int_a^b f(x)\,\mathrm{d}x\right)\left(\int_a^b g(x)\,\mathrm{d}x\right) \leqslant (b-a)\int_a^b f(x)g(x)\,\mathrm{d}x.$$

If we divide by $(b-a)^2$ we find

$$\left(\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right)\left(\frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x\right) \leqslant \frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x,$$

or equivalently that

$$(\text{the mean of } f) \times (\text{the mean of } g) \leqslant (\text{the mean of } fg)$$

when $f$ and $g$ are increasing. This therefore is a continuous version of Chebyshev's sum inequality which appeared in #430.