Solution (#1281) For a positive integer n, we define step functions ϕ_n and ψ_n on $[0,a)$ by

$$\begin{aligned}\phi_n(x) &= \left(\frac{ia}{n}\right)^k \qquad \text{for } \frac{ia}{n} \leqslant x < \frac{(i+1)a}{n} \qquad i = 0,1,2,\ldots,n-1;\\ \psi_n(x) &= \left(\frac{(i+1)a}{n}\right)^k \qquad \text{for } \frac{ia}{n} \leqslant x < \frac{(i+1)a}{n} \qquad i = 0,1,2,\ldots,n-1,\end{aligned}$$

noting

$$\phi_n(x) \leqslant x^k \leqslant \psi_n(x) \qquad \text{for } 0 \leqslant x < a.$$

Recall from #246 we have that

$$\sum_{i=0}^{n-1} i^k = \frac{(n-1)^{k+1}}{k+1} + E_k(n-1)$$

where $E_k(n)$ is a polynomial in n of degree at most k. Deduce as $n \to \infty$ that

$$\int_0^a \phi_n(x)\,\mathrm{d}x \to \frac{a^{k+1}}{k+1} \qquad \text{and} \qquad \int_0^a \psi_n(x)\,\mathrm{d}x \to \frac{a^{k+1}}{k+1}.$$

For the last part use Proposition 5.12(c).