Solution (#1291) We will define two functions $g(x)$ and $h(x)$ such that

$$\lim_{R\to\infty}\int_{-R}^{R} g(x)\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} h(x)\,\mathrm{d}x = 1.$$

That the two functions are both integrable relies on the convergence of the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}+\cdots = 1.$$

Firstly we define $g(x)$ for $x \geqslant 0$ to equal

$$\frac{1}{4} \quad \text{on} \quad [0,1), \qquad \frac{1}{8} \quad \text{on} \quad [1,2), \qquad \frac{1}{16} \quad \text{on} \quad [2,3), \qquad \frac{1}{32} \quad \text{on} \quad [3,4),\ldots$$

and so on. We then extend $g(x)$ to the negative real line by requiring $g(x)$ to be even. Note that

$$\lim_{R\to\infty}\int_{-R}^{R} g(x)\,\mathrm{d}x = 2\lim_{R\to\infty}\int_{0}^{R} g(x)\,\mathrm{d}x = 2\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right) = 1.$$

Note that $g(x) > 0$ for all x.

We then define $h(x)$ to equal

$$1 \quad \text{on} \quad \left(0,\frac{1}{2}\right), \qquad 2 \quad \text{on} \quad \left(1,1\frac{1}{8}\right), \qquad 4 \quad \text{on} \quad \left(2,2\frac{1}{32}\right), \qquad 8 \quad \text{on} \quad \left(3,3\frac{1}{128}\right),\ldots.$$

and so on, and otherwise to equal 0. Note that $h(x) \geqslant 0$ for all x and that

$$\begin{aligned}\lim_{R\to\infty}\int_{-R}^{R} h(x)\,\mathrm{d}x &= 1\times\frac{1}{2}+2\times\frac{1}{8}+4\times\frac{1}{32}+\cdots \\ &= \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = 1.\end{aligned}$$

Note in particular that $h(x)$ is unbounded having each of $1,2,4,8,\ldots$ in its range.

We then have that $f(x) = g(x)+h(x) > 0$ for all x, that $f(x)$ is unbounded and

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\,\mathrm{d}x = 2$$

exists.