Solution (#1301) Say that $y(x)$ is a differentiable function on $\mathbb{R}$ satisfying

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y, \qquad y(0) = 1.$$

If we set $z(x) = y(x)e^{-x}$ then by the product rule

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}e^{-x} - ye^{-x} = ye^{-x} - ye^{-x} = 0.$$

It therefore follows that z is constant; as $z(0) = y(0)e^{-0} = 1$ then that constant is 1 and we have

$$y(x)e^{-x} = 1 \qquad \text{for all } x.$$

Hence

$$y(x) = e^{x} \qquad \text{for all } x.$$