Solution (#1310) For $x \geqslant -1$, define

$$f(x) = (1+x)^{\alpha} - 1 - \alpha x.$$

Note that

$$f'(x) = \alpha(1+x)^{\alpha-1} - \alpha.$$

Note that $f(0) = 0$ and that if $x > 0$ then $f'(x) > 0$. Hence

$$f(x) = f(x) - f(0) = \int_0^x f'(t)\,\mathrm{d}t \geqslant 0$$

and we have proven Bernoulli's Inequality for $x \geqslant 0$.

Now suppose that $-1 < x < 0$. As $f(-1) = \alpha - 1$ and $f'(x) < 0$ for $-1 < x < 0$ then

$$f(0) - f(x) = \int_x^0 f'(t)\,\mathrm{d}t \leqslant 0.$$

For $-1 < x < 0$ we then also have

$$f(x) \geqslant f(0) = 0$$

and we have proven Bernoulli's Inequality for $-1 \leqslant x \leqslant 0$.